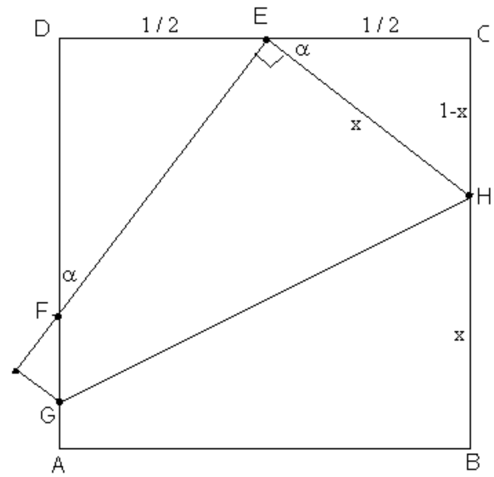
Leap Frog 2004 Solutions

1. Each side of square $ABCD$ has length 1. The square is folded along crease GH in such a way that vertex B is placed at the midpoint E of side CD. The folding causes side AB to meet side CD at point F. What is the distance from D to F?

Solution:



Let x be the length of HE. Since HE where BH is folded to $x = BH$ as well, so $CH = 1 - x$. By the Pythagorean theorem, $x^2 = (1/2)^2 + (1-x)^2$, which implies $x = 5/8$, $CH = 1 - x = 3/8$. Triangles EDF and HCE are similar, so $\dfrac{DF}{DE} = \dfrac{CE}{CH}$, that is, $\dfrac{DF}{1/2} = \dfrac{1/2}{3/8}$, so $DF = 2/3$.

2. A polynomial $P(x)$ satisfies the identity $x^5 - \dfrac{1}{x^5} = P(x - \dfrac{1}{x})$. Write out $P(x)$ with terms in order of decreasing degree.

Solution: From the equations

$$\begin{aligned}
(x-\tfrac{1}{x})^5 &= x^5 - 5x^3 + 10x - \tfrac{10}{x} + \tfrac{5}{x^3} - \tfrac{1}{x^5}\\
(x-\tfrac{1}{x})^3 &= x^3 - 3x + \tfrac{3}{x} - \tfrac{1}{x^3}\\
(x-\tfrac{1}{x}) &= x - \tfrac{1}{x}
\end{aligned}$$

it is clear that $x^5 - \frac{1}{x^5} = (x - \frac{1}{x})^5 + 5(x - \frac{1}{x})^3 + 5(x - \frac{1}{x})$. $P(x) = x^5 + 5x^3 + 5x$.

3. Line up 10 apples in a row. A set of these apples is called "unfriendly" if it contains no two adjacent apples. How many unfriendly sets of 3 apples are there?

Solution: The answer is $\dbinom{8}{3} = 56$. To see this, imagine 8 apples in a row. For any chosen set of 3 of them, put a new apple to the right of each of the first two in the set, and your chosen set of 3 will be an unfriendly set of 3 apples out of a row of 10.

4. Assuming the pattern in this triangle of numbers is continued on, what will the sum of the numbers in the tenth row be?

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 3 & & 5 & & \\ & 7 & & 9 & & 11 & \\ 13 & & 15 & & 17 & & 19 \end{array}$$

Solution: Look at the sums of the 4 displayed rows: 1,8,27,64; the pattern seems to be that the n^{th} row sums to n^3, so the tenth row should sum to 1000. Let's prove this.

The number of entries in the first n rows is $1 + 2 + \cdots + n = n(n+1)/2$, and the sum of the first k odd numbers is $1 + 3 + \cdots + (2k-1) = k^2$. Therefore the sum of the entries in the first n rows is the sum of the first $n(n+1)/2$ odd numbers, $[n(n+1)/2]^2$. Likewise, the sum of the entries in the first $n-1$ rows is the sum of the first $(n-1)n/2$ odd numbers, $[(n-1)n/2]^2$. Thus the sum of the entries in row n is $[n(n+1)/2]^2 - [(n-1)n/2]^2 = n^3$.

5. The sides of a triangle are consecutive integers, and the largest angle is twice the smallest angle. What is length of the shortest side?

Solution: Say the sides are $n, n+1, n+2$, and the corresponding angles are $\theta, 180^o - 3\theta, 2\theta$. The law of sines tells us that $\dfrac{n+2}{n} = \dfrac{\sin 2\theta}{\sin\theta} = 2\cos\theta$, so $n = \dfrac{2}{2\cos\theta - 1}$.

The law of sines also tells us that $\dfrac{n+1}{n} = \dfrac{\sin 3\theta}{\sin\theta} = 3 - 4\sin^2\theta = 4\cos^2\theta - 1$, so $n = \dfrac{1}{4\cos^2\theta - 2}$.

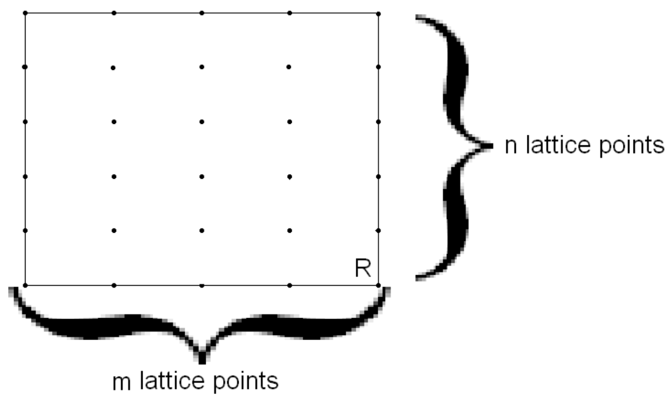
Now equate the two expressions for n: $\cos\theta - \frac{1}{2} = 4\cos^2\theta - 2$, which factors as $(4\cos\theta - 3)(2\cos\theta + 1) = 0$. Now $\cos\theta$ cannot be negative (the smallest angle in a triangle cannot be obtuse), so $\cos\theta = \frac{3}{4}$. Therefore $n = \dfrac{2}{2\cos\theta - 1} = 4$.

6. HATBOX is a 6-digit number – each letter represents a different decimal digit. If 9·HATBOX = 4·BOXHAT, what number is HATBOX?

Solution: Let x be the number HAT, and y the number BOX. Then $9000x + 9y = 4000y + 4x$, so $8996x = 3991y$. Now $8996 = 13 \cdot 692$ and $3991 = 13 \cdot 307$, so $692x = 307y$. One solution (and the only 6-digit solution), is HATBOX = 307,692.

7. 7. A *lattice point* is a point (x, y) in the Cartesian plane where x and y are both integers. A rectangle R has its vertices at lattice points and its sides parallel to the coordinate axes. The sides of R contain a total of 62 lattice points, and the interior contains 100 lattice points. Compute the area of the rectangle R.

Solution:



Notice that any rectangle consisting of m lattice points by n lattice points has area $A = (m-1)(n-1)$, $I = (m-2)(n-2)$ lattice points in the interior, and $B = 2m + 2n - 4$ lattice points on the boundary. Therefore $A = mn - m - n + 1 = (mn - 2m - 2n + 4) + \frac{1}{2}(2m + 2n - 4) - 1 = I + \frac{1}{2}B - 1$. In this case the area is $100 + \frac{1}{2} \cdot 62 - 1 = 130$.

8. There are 5 suspects in the Bogarde heist. One is guilty. Each suspect makes a statement.

Emmanuel Ravelli: Either Baravelli or Chicolini did it.

Baravelli: Neither Chicolini nor Tony is guilty.

Chicolini: Emmanuel Ravelli and Baravelli are both lying.

Fiorello: Exactly one of Emmanuel Ravelli and Baravelli is telling the truth.

Tony: Fiorello is wrong.

We know that at least 3 of these statements are true. Who is the guilty party?

Solution: If Chicolini were telling the truth, then Emmanuel Ravelli, Baravelli and Fiorello would all be lying, which is impossible. If Fiorello were telling the truth, then one of Emmanuel Ravelli or Baravelli would by lying, as would Chicolini and Tony. This is also impossible. Therefore Emmanuel Ravelli, Baravelli and Tony must all be telling the truth. But Emmanuel Ravelli's and Baravelli's statements together imply that Baravelli is the culprit.

(The characters in this problem are the parts played by Chico Marx in the films Animal Crackers, Horsefeathers, Duck Soup, Night at the Opera and Day at the Races, respectively.)