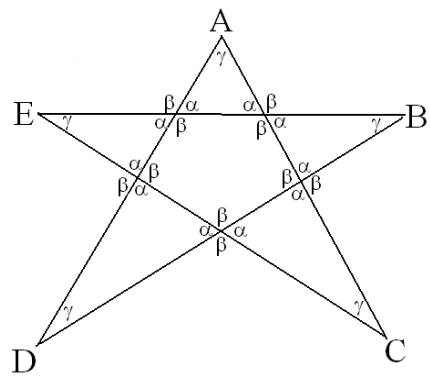
Leapfrog 2007 Solutions

1. In the figure below, find the sum of angles A, B, C, D and E.



Solution: Let a be the sum of the angles labeled α, b the sum of the angles labeled β, and c the sum of the angles labeled γ. Then $a + b = 5 \cdot 360^o = 1800^o$. But $\frac{1}{2}b$ is the sum of the angles of a pentagon, so $b = 2 \cdot 540^o = 1080^o$, so $a = 720^o$. Finally, $a + c = 5 \cdot 180^o = 900^o$, so $c = 180^o$.

2. Suppose that x, y, z are positive numbers,and

$$(x + y + z)^x = 4, \qquad (x + y + z)^y = 2, \qquad \text{and } (x + y + z)^z = 32.$$

Evaluate z (as a fraction in lowest terms).

Solution: Multiply the 3 equations together; we get$(x + y + z)^{x+y+z} = 256 = 4^4$, so $x + y + z = 4$. Therefore $4^z = 32$, so $z = 5/2$.

3. Flip a coin 6 times. What is the probability that no 2 consective tosses are heads?

Solution: Let's count the number of ways n tosses can have no 2 consecutive heads, for $n = 1, 2, 3, \ldots$. Call this number f_n.

For $n = 1$, there are 2: H and T. $f_1 = 2$.

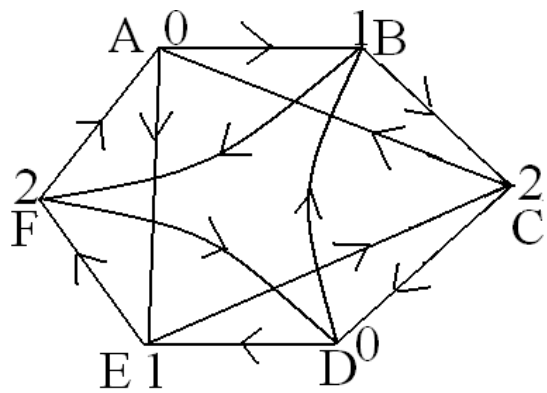
For $n = 2$, there are 3, HT, TH and TT. $f_2 = 3$.

For $n = 3$, if the first coin is T, then there are 2 – append any of the sequences from $n = 2$ after the T. If the first coin is H, the next must be T, and then append any of the sequence from $n = 1$. $f_3 = f_2 + f_1 = 5$.

Applying the reasoning in the case $n = 3$ to any $n > 2$, we see that $f_n = f_{n-1} + f_{n-2}$. Therefore $f_4 = 5 + 3 = 8$, $f_5 = 8 + 5 = 13$, and $f_6 = 13 + 8 = 21$.

When you flip a coin 6 times, the *probability* that no 2 consecutive tosses are heads is $f_6/2^6 = 21/64$.

4. An ant moves from labeled point to labeled point along the curves in the direction of the arrows in the diagram below. At time 0, it is at vertex A. At time t (for $t = 0, 1, 2, \ldots$), he chooses at random one of the two curves whose arrow leads away from the point he is on, and travels along that curve until he arrives at the next lableled point at time $t + 1$. What is the probability that he is back at point A at time $t = 1000$?



Solution: Notice that from each vertex marked 0, the ant must move to a vertex marked 1; from each vertex marked 1, the ant must move to a vertex marked 2; from each vertex marked 2, the ant must move to a vertex marked 0. Since the ant is at a vertex marked 0 on day 0, he will be at vertices marked 0 only on days divisible by 3. The required probability is 0.

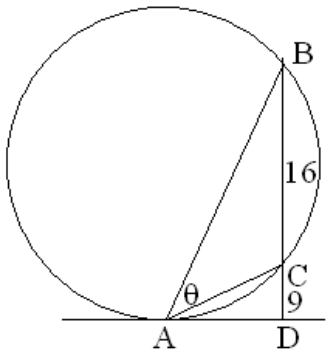
5. Evaluate exactly:

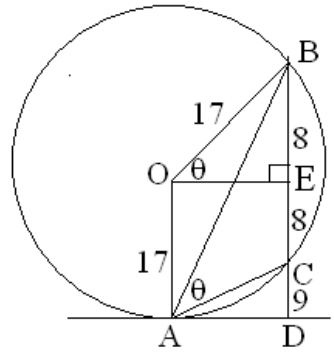
$$1^2\left(\frac{1}{1^2}-\frac{1}{2^2}\right) + (1^2{+}2^2)\left(\frac{1}{2^2}-\frac{1}{3^2}\right) + (1^2{+}2^2{+}3^2)\left(\frac{1}{3^2}-\frac{1}{4^2}\right) + (1^2{+}2^2{+}3^2{+}4^2)\left(\frac{1}{4^2}-\frac{1}{5^2}\right) +$$

$$\cdots + (1^2{+}2^2{+}3^2{+}\cdots{+}99^2)\left(\frac{1}{99^2}-\frac{1}{100^2}\right) + (1^2{+}2^2{+}3^2{+}\cdots{+}100^2)\left(\frac{1}{100^2}\right).$$

Solution: For $i = 1, \ldots, 100$, the fraction $1/i^2$ is multiplied by the sum $1^2 + 2^2 + \cdots + i^2$ in one term, and by $-(1^2 + 2^2 + \cdots + (i-1)^2)$ in another term, so the sum of the two terms involving $1/i^2$ is $\frac{1}{i^2} \cdot i^2 = 1$. The overall sum is exactly 100.

6. In the figure below, circle $\mathcal{K}$ contains points A, B, C, line AD is tangent to $\mathcal{K}$, and line BC is perpendicular to line AD. If $BC = 16$ and $CD = 9$, compute $\sin(\angle BAC)$.





Solution: Drop a perpendicular OE from O to BC. This makes E the midpoint of chord BC, so $BE = EC = 8$. Quadrilateral $OEAD$ is a rectangle, so the circle has radius $OA = EC + CD = 17$. Therefore radius $OB = 17$, too. Now $\angle BOE$ is a central angle which subtends half of arc BC, and $\angle BAC$ is inscribed in arc BC, so $\angle BAC = \angle BOE$. Therefore $\sin(\angle BAC) = \sin(\angle BOE) = 8/17$.

7. For any natural number n, $\sigma(n)$ is defined to be the sum of all of the divisors of n. For example, since the divisors of 12 are 1,2,3,4,6,12, $f(12) = 28$.

Suppose p, q, r are three different prime numbers. Find a formula for $\sigma(pqr)$, in factored form.

Solution: The divisors of pqr are $1, p, q, r, pq, qr, pr, pqr$, so $\sigma(pqr) = 1 + p + q + r + pq + qr + pr + pqr = (1 + p)(1 + q)(1 + r)$.

8. Four notorious thieves named Rickey, Lou, Maury and Ty are accused of theft at El Segundo military base. Each makes a statement.

Rickey :	Lou is guilty.
Lou :	Maury is guilty.
Maury :	Ty is guilty.
Ty :	Either Rickey or Maury is guilty, but not both.

Each innocent suspect has told the truth, each guilty suspect has lied. Name the culprits.

Solution: Notice that Rickey is lying, and therefore guilty, if and only if Lou is inoocent, and Lou is innocent if and only if Maury is gulity. Therefore Ty's statement cannot be true, so Ty is guilty. Therefore Maury is innocent, so Lou is guilty, so Rickey is innocent. The culprits are Lou and Ty.