2015 MATH FIELD DAY
HUDDLE PROBLEMS AND SOLUTIONS

Problem 1. Let a and b be real numbers not equal to 0, 1, or each other. Let x and y be nonzero real numbers such that

$$\frac{x}{a} + \frac{y}{a-1} = 1 \qquad \text{and} \qquad \frac{x}{b} + \frac{y}{b-1} = 1.$$

Simplify the expression

$$\frac{y}{x}\left(\frac{a}{a-1}\right)\left(\frac{b}{b-1}\right).$$

Solution. Re-write the equations as

$$x(a-1) + ya = a(a-1) \tag{1}$$

$$x(b-1) + yb = b(b-1) \tag{2}$$

Subtracting $(a-1)$ times equation (2) from $(b-1)$ times equation (1) gives

$$ya(b-1) - yb(a-1) = (a-1)(b-1)(a-b),$$

simplifying to $y = -(a-1)(b-1)$. Similarly, subtracting a times equation (2) from b times equation (1) simplifies to give $x = ab$. Thus

$$\frac{y}{x}\left(\frac{a}{a-1}\right)\left(\frac{b}{b-1}\right) = \frac{-(a-1)(b-1)}{ab} \cdot \frac{a}{a-1} \cdot \frac{b}{b-1} = \boxed{-1.}$$

□

Problem 2. Find the only six-digit number $861abc$ which is divisible by 210 and has no repeated digit.

Solution. To be divisible by 210 is tantamount to being divisble by 2, 3, 5, and 7. Being divisible by 2 and 5 forces $c = 0$. To be divisible by 3 requires the sum of the digits $(8+6+1+a+b+0 = 15+a+b)$ to be divisible by 3, so this forces $a+b$ to be a multiple of 3. Since 861,000 is a multiple of 7, for $861ab0$ to be a multiple of 7 it suffices for the three-digit number $ab0$ to be a multiple of 7, and since $a+b$ is a multiple of 3, we know $ab0$ must also be a multiple of 210 (we could have also just verified that 861000 was a multiple of 210 immediately, with the same conclusion). There are only five such possibilities:

$$861000 \qquad 861210 \qquad \boxed{861420} \qquad 861630 \qquad 861840$$

The solution, the boxed number in the list, is the only one of these with no repeated digit. □

Problem 3. The function

$$\sin\left(\frac{x}{3}+\frac{1}{4}\right)+\sec\left(\frac{x}{5}+\frac{1}{6}\right)+\tan\left(\frac{x}{7}+\frac{1}{8}\right)$$

is periodic. What is its period (in radians)?

Solution. The periods of the respective summands are 6π, 10π, and 7π. The period of their sum is the least common multiple of these periods, which is $\boxed{210\pi.}$

□

Problem 4. Mathball is a game for three teams which results in one winner and two losers. A mathball tournament consists of $3^{10} = 59,049$ teams competing against each other. In each round, the teams are grouped into sets of three and play a round of mathball, with the winning team moving on to the next round, and the losing two teams being eliminated. This continues until only one team, the champion, remains. How many total matches of mathball are played in such a tournament?

Solution. Each match in the tournament eliminates two teams. To eliminate all but the winning team, we need to eliminate 59,048 teams. This takes $\frac{59048}{2} = \boxed{29524}$ matches.

Alternatively, we can attack this problem computationally by reasoning that there will be $\frac{3^{10}}{3} = 3^9$ total matches in the first round, then the 3^9 winners will be organized into 3^8 matches in the second, and so on, down to the last round having 1 match to determine the winnter from the remaining 3^1 teams. There are thus

$$3^9 + 3^8 + 3^7 + \cdots + 3^2 + 3^1 + 3^0$$

total matches. We can then either compute these terms by hand, or apply the formula for a finite geometric series to get the same answer as above. □