2015 MATH FIELD DAY
LEAPFROG PROBLEMS AND SOLUTIONS

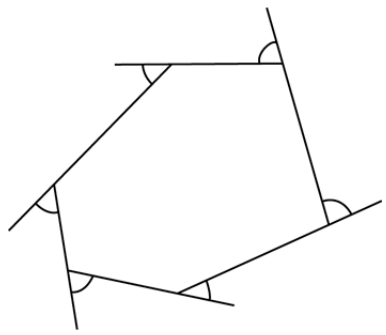
Problem 1. Let i denote a square root of -1, and consider the sequence of numbers defined by $y_0 = 1$, $y_1 = 5$, and $y_n = iy_{n-1} + y_{n-2}$ for $n \geq 2$. Evaluate $\dfrac{y_{2015}}{y_{2009}}$.

Solution. Actually, for any n, we have

$$y_n = iy_{n-1} + y_{n-2} = i(iy_{n-2} + y_{n-3}) + y_{n-2} = iy_{n-3},$$

so $y_n = iy_{n-3} = i(iy_{n-6}) = -y_{n-6}$. So $\frac{y_n}{y_{n-6}} = \boxed{-1.}$ □

Problem 2. In a polygon, the exterior angles are the angles supplementary to the interior angles, as in the figure below.



In a given regular n-gon, each interior angle measures exactly 7 times each exterior angle. Find n.

Solution. The interior angle measure of a regular n-gon is $\frac{180(n-2)}{n}$, and the exterior angle measure is $\frac{360}{n}$. We are given that

$$7 = \frac{180 \cdot \frac{n-2}{n}}{\frac{360}{n}} = \frac{n-2}{2},$$

and solving gives $\boxed{n = 16.}$ □

Problem 3. Let $r(x)$ be the remainder after diving x^{2015} by $x^2 - x$. Find $r(2)$.

Solution. Write $x^{2015} = q(x)(x^2 - x) + r(x)$. Then $r(x)$ is a linear polynomial, and evaluating that equality at $x = 0$ and $x = 1$ respectively gives $r(0) = 0$ and $r(1) = 1$. There is only one linear polynomial $r(x)$ satisfying $r(0) = 0$ and $r(1) = 1$ – namely, the polynomial $r(x) = x$. Thus $\boxed{r(2) = 2.}$ □

Problem 4. An arithmetic sequence of positive integers contains the numbers 6 and 122, though 122 is not the term directly after 6. There are no perfect squares in the sequence between the terms 6 and 122. What is the next number in the sequence after 122?

Solution. The common difference of the arithmetic sequence must be a divisor of $122-6 = 116$, and is not 116 itself, since 6 and 122 are not successive terms in the sequence. That leaves the possible common differences of 1, 2, 4, 29, and 58. The sequence proceeding from 6 with common difference 1 hits the perfect square number 16 (among others), as does the sequence with common difference of 2. The sequence proceeding from 6 with common difference either 29 or 58 hits the perfect square 64. The only remaining possibility is a common difference of 4, which does indeed avoid all perfect squares between 6 and 122 (every even square is a multiple of 4, and no terms in this sequence would be). Since we have deduced that the common difference must be 4, the term after 122 must be $\boxed{126.}$

□

Problem 5. Find the smallest positive real solution x (in radians) to the equality

$$\sqrt{3}\cos(x) - \sin(x) = \sqrt{2}.$$

Solution. The left-hand side is simply $2\sin(\pi/3 - x)$, and so we need only solve

$$\sin(\pi/3 - x) = \frac{\sqrt{2}}{2}.$$

This happens first when $\frac{\pi}{3} - x = \frac{\pi}{4}$, and so the smallest positive such value of x is $\boxed{x = \frac{\pi}{12}.}$ □

Problem 6. Find the largest integer $m < 45$ such that m^m divides evenly into 45^{45}.

Solution. First, we note

$$45^{45} = 9^{45}5^{45} = 3^{90}5^{45},$$

so m can only have 3's and 5's in its prime factorization. If we write $m = 3^a5^b$, then

$$m^m = (3^a5^b)^{3^a5^b} = 3^{a3^a5^b}5^{b3^a5^b}.$$

For this to go into 45^{45}, we need to have $a3^a5^b \leq 90$ and $b3^a5^b \leq 45$. This second inequality forces $b < 2$. If $b = 1$, then we can take $a = 1$ ($a = 2$ is not possible since $m < 45$), giving $m = 15$. But if $b = 0$, we can satisfy both inequalities with $a = 3$, giving $\boxed{m = 27.}$ This is thus the largest possible such m. □

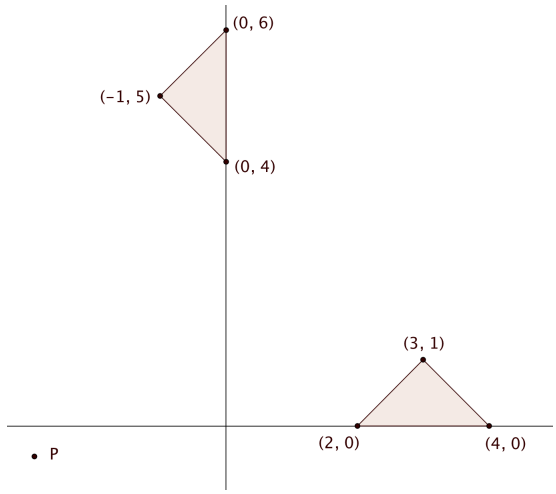
Problem 7. Find the radius of a circle inscribed in a triangle with two sides of length 17 and one side of length 16.

Solution. There are many possible solutions. Here's one via an area calculation: Dropping the altitude to the base of this isosceles triangle, we compute via the Pythagorean Theorem that the height is 15, and so the area of the triangle is 120. Alternatively, dropping three altitudes (of length equal to the desired radius r) from the inscribed circle to the three sides partitions the triangle into three smaller triangles, with total area

$$\frac{16r}{2} + \frac{17r}{2} + \frac{17r}{2} = 25r.$$

Since $25r = 120$, we have $\boxed{r = \frac{24}{5}.}$ □

Problem 8. A rotation of the plane around a point P takes the triangle with vertices $(2, 0)$, $(4, 0)$, and $(3, 1)$ to the triangle with vertices $(0, 4)$, $(0, 6)$, and $(-1, 5)$. Find the coordinates of P.



Note: Not drawn to scale/accuracy.

Solution. The point P must be equidistant from $(2, 0)$ and $(0, 4)$, so must lie on the perpendicular bisector of the segment joining them. Since the line through them has equation $y = -2x + 4$, the perpendicular bisector has the negative reciprocal slope $\frac{1}{2}$ and goes through the midpoint $(1, 2)$, giving us the equation $y = \frac{1}{2}(x-1)+2$. Similarly, P must also be equidistant from $(4, 0)$ and $(0, 6)$, and repeating the reasoning gives us the perpendicular bisector $y = \frac{2}{3}(x - 2) + 3$. Finding the point of intersection of these two perpendicular bisectors gives us the point $\boxed{P = (-1, 1).}$ $\square$