

Math Field Day 2016  
Mad Hatter A

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Warmup 1.

If  $x = 1 + \frac{1}{1 + \frac{1}{1+1}}$ ,  
evaluate  $\frac{1}{\frac{1}{x-1} - 1}$ .

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Warmup 2.

Simplify

$$\frac{2016^2 - 2016 \times 32 + 256}{10^6}$$

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Warmup 3.

$$2^2 \cdot 2^3 \cdot \dots \cdot 2^{10} = 4^x.$$

Evaluate  $x$ .

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Warmup 4.

Is the sum of 56 consecutive integers

- (a) always odd,
- (b) always even, or
- (c) sometimes odd, sometimes even?

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A1. I am twice as old as I was when I was 3 years older than I was when I was one-third my present age. How old am I?

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A2. Find the sum of the angles  $\theta$  in radians with  $0 < \theta < 2\pi$  and  $\sin^2 \theta = 1/3$ .

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A3.

A rhombus has area 40 and a diagonal of length 8. What is the length of the other diagonal?

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A4.

Suppose that

$$2016x + 2015y = 6047,$$

$$2015x + 2016y = 6046.$$

Find  $\frac{x-y}{x+y}$ .

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A5.

Suppose  $a \circ b = a^2 + b^2 - ab$ .

Find the value of

$$(1 \circ 2) \circ (3 \circ 4).$$

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A6.

$$\log_2(2016!) - \log_2(2015!) = 5 + \log_2 b.$$

$$b = ?$$

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A7.  $s$  and  $t$  are the roots of

$$x^2 - 8x + 3 = 0.$$

$s + 1$  and  $t + 1$  are the roots of

$$x^2 + ax + b = 0.$$

$$b = ?$$

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A8.

The parabola  $y = x^2 + bx + c$  has vertex  $(1, -4)$ , and it intersects the  $x$ -axis at  $x = r$  and  $x = s$ .

Find the value of  $|r - s|$ .

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A9.

How many ways can five people sit at a round table, if rotated seatings are considered the same?

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A10. Consider the function

$$f(x) = kx + 2016,$$

where  $k \neq 0$ .

If  $f(f(2016)) = 2016$ , find  $k$ .

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A11. A right triangle has a  $30^\circ$  angle, and the square of its perimeter is  $m + 54\sqrt{3}$ , where  $m$  is an integer. The length of the hypotenuse is an integer. What integer is it?

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A12.  
Find the circumference of the circle that contains the points  $(0,0)$ ,  $(0,8)$  and  $(6,8)$ .

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A13.  $(x + \frac{1}{x}) + (y + \frac{1}{y}) = 1.$   
 $(x^3 + \frac{1}{x^3}) + (y^3 + \frac{1}{y^3}) = 16.$   
 $(x + \frac{1}{x})^3 + (y + \frac{1}{y})^3 = ?$

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A14. Write this number in the simplest form possible.

$$4^{1/\log_5 2} + 2^{1/\log_9 4}$$

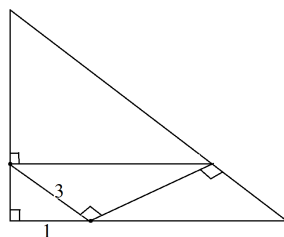
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A15.  
How many lines contain two of the vertices of a cube, but none of the edges?

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A16.  
 $a$ ,  $b$ , and  $c$  are integers.  
 $x^4 + ax^3 + bx^2 + cx + 25$  has four different rational roots.  
 $a + b + c = ?$

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A17. Find the length of the hypotenuse of the outer triangle.



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A18.  
Mary has as many brothers as sisters, but each of her brothers has twice as many sisters as brothers. How many brothers does Mary have?

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A19.  
Solve for  $x$ :  
 $4^x = 2^{x+2} + 2^{x+1} + 16.$

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A20.  
Compute the area of the figure bounded by the graph of the equation  $|x| + |y| = 2$ .

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A21. For all positive integers  $n$ ,

$$\sum_{i=1}^n b_i = \frac{n-1}{n+1}.$$

$b_5 = ?$

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A22.  $a$ ,  $b$ ,  $c$ ,  $d$  are real numbers,  $a > b$  and  $c > d$ .  
When is it true that  $ac + bd > ad + bc$ ?  
(i) Always.  
(ii) Never.  
(iii) Sometimes, but not always.

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A23.  
Evaluate  $\log_3(1 + 2n)$ , where  $n = 1 + 3 + 9 + 27 + \dots + 3^{2016}$ .

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A24. Every point on the line  $y = -2x$  is equidistant from the points  $(k, 2)$  and  $(-k, -2)$ . Find  $k$ .

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A25.  
Evaluate  $303^2 - 298^2$ .

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A26. Evaluate  
$$\arctan \left( \frac{1}{2016} + \sum_{n=1}^{2015} \frac{1}{n(n+1)} \right).$$

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A27. The lengths of the parallel sides of a trapezoid differ by 2. Both of the other sides have length  $\sqrt{5}$ . The trapezoid has area 9. What is the length of the shorter parallel side?

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A28.  
Suppose that  $b$  is a real number such that  $x^2 + bx + 14$  has one root that is twice the other. Find  $b^2$ .



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A29.  
At a party with eleven people, each pair of people shakes hands just once. How many handshakes are there?



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A30.  
Solve the equation  $\log_{\sqrt{x}}(5\sqrt{x} + 6) = 2$  for  $x$ .



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A31.  
Let  $f(x) = 1 + x + x^2 + x^4 + x^8 + x^{16}$  for all  $x$ .  
Solve for  $y$ :  
 $f(y - 1) = f(1 - y) + 6$ .



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A32. How many real numbers  $x$  are there such that  $\frac{16}{1 + 3^x}$  is an integer?



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A33.  $\cos 2\theta = 5 \cos \theta - 3$ .  
 $\cos \theta = ?$



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A34.  
Two positive integers differ by more than 1. Their squares differ by 65. Find the larger integer.



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A35.  
 $x^{\log_3 x} = 81$   
 $0 < x < 1$   
 $x = ?$



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A36.  
A rectangle has sides of lengths 3 and 4. Find the distance from a vertex to the diagonal through two other vertices.



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A37.  
Find the smallest solution to the equation  
 $|2000 - |x - 2016|| = 16$ .



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A38. Evaluate  
 $\ln(\sin(2010^\circ)) + \ln(\sec(2010^\circ)) + \ln(\cot(2010^\circ))$ .



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A39.

2	x	
4		

The cells in the magic square above are to be filled with the numbers from 1 through 9, a different number in each cell. The sum of the 3 entries in each row, each column and each diagonal are to be the same.  $x = ?$



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A40. Andrew and Camille play a game. Andrew starts by choosing a number from 1 through 9, and the players alternate choosing a number from 1 through 9 not previously chosen. The winner is the first player whose choices include three numbers which sum to 15. If when all of the number from 1 through 9 have been chosen neither player has chosen three which sum to 15, the game ends in a draw.  
Andrew begins the game by choosing 2. What number must Camille choosing to prevent Andrew from winning?



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B1. I am twice as old as I was when I was 3 years older than I was when I was one-third my present age. How old am I?

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B2. For what real number  $k$  does the line  $x + y = 11$  contain the midpoint of the points  $(1, 3k)$  and  $(k, 5)$ ?

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B3.

A rhombus has area 40 and a diagonal of length 8. What is the length of the other diagonal?

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Suppose that

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Suppose  $a \circ b = a^2 + b^2 - ab$ .

Find the value of

$$(1 \circ 2) \circ (3 \circ 4).$$

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B6.

Find the sum of the zeros of  $(x - 3)^4 - 29(x - 3)^2 + 100$ .

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B7.  $s$  and  $t$  are the roots of

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Simplify  $f(f(x))$ ,  
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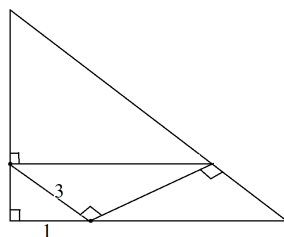
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B23. Find the smallest positive integer  $b$  such that  $x^2 + bx + 13$  has at least one real root.

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B24. Every point on the line  $y = -2x$  is equidistant from the points  $(k, 2)$  and  $(-k, -2)$ . Find  $k$ .

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B25.  
Evaluate  $303^2 - 298^2$ .

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B26. Randomly choose two different integers from 10 through 20. What is the probability that their sum is odd?

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At a party with eleven people, each pair of people shakes hands just once. How many handshakes are there?



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Simplify 
$$\frac{b^{3/2} - 3b + 2}{b - 2\sqrt{b} - 2}$$
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Let  $f(x) = 1 + x + x^2 + x^4 + x^8 + x^{16}$  for all  $x$ .  
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B33. Choose a number from 1 through 100 at random. What is the probability that your chosen number is either even or a multiple of 3?



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B34.  
Two positive integers differ by more than 1. Their squares differ by 65. Find the larger integer.



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B35.  $a, b, c$  are integers.  
 $a > 1$ .  
 $(a + b\sqrt{7})^2 = c + 6\sqrt{7}$ .  
 $c = ?$



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B36.  
A rectangle has sides of lengths 3 and 4. Find the distance from a vertex to the diagonal through two other vertices.



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B37.  
Find the smallest solution to the equation  
 $|2000 - |x - 2016|| = 16$ .



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B38. Suppose that for some integer  $x$ ,  $[3(x + 8)]^2$  is the 6-digit integer  $149m69$ .  
 $m = ?$



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B39.

2	$x$	
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