2017 MATH FIELD DAY
HUDDLE PROBLEMS AND SOLUTIONS

Problem 1. Find a real number x satisfying the following equation:

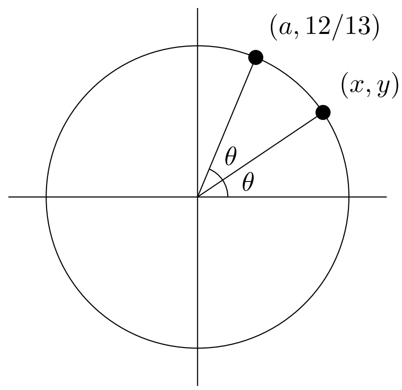
$$x^5 - \sqrt{1-(x-4)^2} = 5^x - \sqrt{x(12-x)-35}$$

Solution. The only possible answer is $\boxed{x = 5.}$ If we rewrite the radicals to obtain the equivalent identity

$$x^5 - \sqrt{-(x-3)(x-5)} = 5^x - \sqrt{-(x-7)(x-5)},$$

we see that both radicals evaluate to 0 at $x = 5$, and certainly x^5 and 5^x are equal when $x = 5$. In fact, this re-writing also shows that $x = 5$ is the only value for which both sides of the expression are even defined, as the left-hand radical is only defined for $3 \le x \le 5$, and the right-hand radical only for $5 \le x \le 7$. □

Problem 2. The figure below shows two points on the unit circle and two equal angles. Evaluate $\dfrac{1}{x^4 - y^4}$.



Solution. We have both $x^2 + y^2 = 1$ and $a^2 + \left(\frac{12}{13}\right)^2 = 1$. The latter gives $a = \frac{5}{13}$, and so the former gives

$$\frac{1}{x^4 - y^4} = \frac{1}{x^2 + y^2} \cdot \frac{1}{x^2 - y^2} = 1 \cdot \frac{1}{\cos^2(\theta) - \sin^2(\theta)} = \frac{1}{\cos(2\theta)} = \frac{1}{5/13} = \boxed{\frac{13}{5}.}$$

□

Problem 3. A Ruth-Aaron pair is a pair of consecutive integers such that the sum of their prime divisors is the same. For example, 77 and 78 form a Ruth-Aaron pair since $77 = 7 \cdot 11$, $78 = 2 \cdot 3 \cdot 13$, and $7 + 11 = 2 + 3 + 13$.

If p and q are prime numbers such that $102p$ and $115q$ form a Ruth-Aaron pair, find $p + q$.

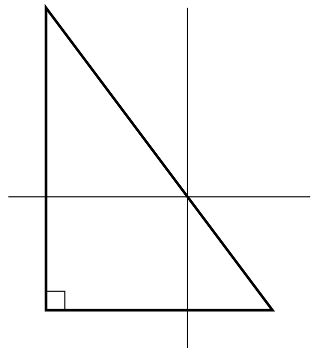
Solution. Let us temporarily assume that p does not divide 102 and q does not divide 115. Then since $102 = 2 \cdot 3 \cdot 17$, the sum of the prime divisors of $102p$ is $2 + 3 + 17 + p = 22 + p$. Similarly, since $115 = 5 \cdot 23$, the sum of the prime divisors of $115q$ is $5 + 23 + q = 28 + q$. For these to be equal, we need $22 + p = 28 + q$, so $p = q + 6$. We also need these two numbers, $115q$ and $102(q + 6)$ to be consecutive in some order, so either

$$115q + 1 = 102(q + 6) \qquad \text{or} \qquad 102(q + 6) + 1 = 115q.$$

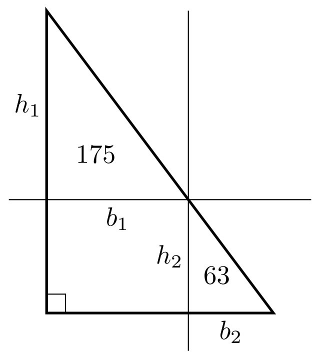
The first simplifies to $13q = 611$ and the second to $13q = 613$. Only the first of these has an integer solution of q, namely, $q = 47$, and so $p = q + 6 = 53$, and so $p + q = 47 + 53 = \boxed{100.}$

To complete the argument by addressing the other cases: If p *were* to divide 102, the sum of the prime divisors of $102p$ would simply be 22, which is less than 28 (and hence $28 + q$ for any prime q). If q were to divide 115, then the sum of the divisors of $115q$ would be 28, and no prime p makes $22 + p$ equal to 28. □

Problem 4. The right triangle below has been cut by two lines parallel to its two legs to divide it into two triangular regions, of respective areas 175 and 63, and a rectangular region of area A. Find A.



Solution. Since the cutting lines are parallel to the two legs, both smaller triangles are right triangles, and similar to the original triangle (and hence each other) by the angle-angle condition. Call the bases and heights of the triangles b_1, b_2, h_1, and h_2 as labeled below:



By the given areas, we see $h_2b_2 = 2 \cdot 63 = 126$ and $h_1b_1 = 2 \cdot 175 = 350$. Further, since the two smaller triangles are similar, we have

$$\frac{h_1}{h_2} = \frac{b_1}{b_2} \qquad \text{or} \qquad h_1b_2 = h_2b_1.$$

Finally, the square of the area of the rectangle is

$$(b_1h_2)^2 = b_1h_2b_1h_2 = (h_2b_2)(h_1b_1) = 2 \cdot 63 \cdot 2 \cdot 175 = 2^2(3^2 \cdot 7)(5^2 \cdot 7) = 2^2 3^3 5^2 7^2.$$

Taking square roots gives the area $A = b_1h_2 = 2 \cdot 3 \cdot 5 \cdot 7 = \boxed{210.}$ □