2017 MATH FIELD DAY
LEAP FROG PROBLEMS AND SOLUTIONS

Problem 1. A rectangle has the following properties:

- Its length and width are positive integers, neither of which is an integer multiple of the other.
- Its area is three times its perimeter.

Find its area.

Solution. Call the length ℓ and width w. The given area-perimeter constraint tells us that $3(2\ell + 2w) = \ell w$, which we can solve for w to get

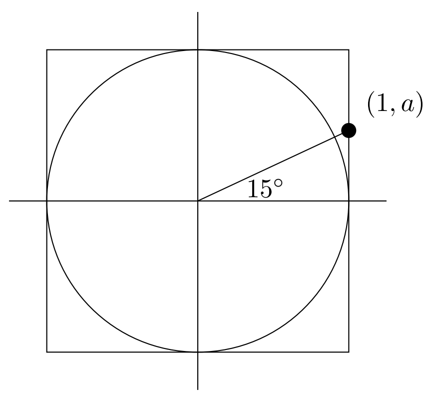
$$w = \frac{6\ell}{\ell - 6} = \frac{6(\ell - 6) + 36}{\ell - 6} = 6 + \frac{36}{\ell - 6}.$$

For w to be an integer, we need $\ell - 6$ to divide evenly into 36, so $\ell - 6$ is 1,2, 3, 4, 6, 9, 12, 18, or 36. We get the following table of possible ℓ's and w's:

ℓ	7	8	9	10	12	15	18	24	42
w	42	24	18	15	12	10	9	8	7

The only pairs in this table which satisfies the property that neither is a multiple of the other consist of 10 and 15 in some order, so the area must be $\boxed{150.}$ □

Problem 2. The figure below shows the unit circle and a point marked on the circumscribed unit square.



What integer is $\left(\frac{1}{a+1} + \frac{1}{a-1}\right)^{-2}$?

Solution. Considering the triangle with vertices at $(0, 0)$, $(1, 0)$, and $(1, a)$, we see that $a = \tan(15)$. Thus

$$\frac{1}{a+1} + \frac{1}{a-1} = \frac{a-1}{a^2-1} + \frac{a+1}{a^2-1} = \frac{2a}{a^2-1} = -\frac{2\tan(15)}{1-\tan^2(15)} = -\tan(30) = -\frac{\sqrt{3}}{3}$$

where in the last step we have used the sum formula for tangent. Raising this value to the -2nd power gives the final answer of $\boxed{3.}$ □

Problem 3. For any real number x, let $\lfloor x \rfloor$ be the greatest integer less than or equal to x. For example, we have $\lfloor 2.19 \rfloor = 2$ and $\lfloor 3 \rfloor = 3$. In effect, $\lfloor x \rfloor$ rounds x down to an integer.

Consider the sequence defined by $a_1 = 2017$ and for $n > 1$,

$$a_n = 1 + n \left\lfloor \frac{a_{n-1} - 1}{n} \right\rfloor.$$

Find a_{2017}.

Solution. The expression

$$n\left\lfloor \frac{x-1}{n} \right\rfloor$$

gives the largest multiple of n less than x. Thus a_2 is one more than the largest even number less than 2017, so $a_2 = 2017$. Similarly, a_3 is one more than the largest multiple of 3 less than 2017, so $a_3 = 2017$. It is clear from this description that $a_i \leq 2017$ for all i, and in particular, $a_{2017} \leq 2017$. Then the smallest multiple of 2017 less than a_{2017} is 0, and so $a_{2017} = \boxed{1.}$

For the curious, the sequence begins

2017, 2017, 2017, 2017, 2017, 2017, 2011, 2010, 2009, 2008, 2001, 1992, 1981, 1977, 1975, 1966, 1953, 1939, 1927, 1920, 1901, 1891, 1871, 1864, 1849, 1826, 1821, 1810, 1793, 1770, 1741, 1737, 1729, 1717, 1701, 1681, 1657, 1629, 1597, 1561, 1561, 1559, 1555, 1549, 1541, 1531, 1519, 1505, 1489, 1471, 1451, 1429, 1405, 1379, 1351, 1321, 1289, 1255, 1219, 1181, 1141, 1099, 1055, 1009, 961, 911, 859, 805, 749, 691, 631, 569, 505, 439, 371, 301, 229, 155, 79, 1, 1, 1 , ...

and continues with $a_n = 1$ for all $n \geq 80$. □

Problem 4. Consider the polynomial

$$(3x+5)^6(x^2+x-2)^4(4+3x)^8 = a_{22}x^{22} + a_{21}x^{21} + a_{20}x^{20} + \cdots + a_2x^2 + a_1x + a_0.$$

of degree 22. Find $a_0 + a_2 + a_4 + \cdots + a_{22}$.

Solution. Call the polynomial $f(x)$. Then $f(1) = a_{22} + a_{21} + a_{20} + \cdots + a_1 + a_0$, and

$$f(-1) = a_{22} - a_{21} + a_{20} - a_{19} + a_{18} - \cdots + a_2 - a_1 + a_0.$$

Adding these two and dividing by 2 gives

$$a_0 + a_2 + a_4 + \cdots + a_{22} = \frac{f(1) + f(-1)}{2}.$$

It remains only to evaluate $f(1) = 0$ (due to the middle factor: $1^2+1-2=0$) and $f(-1) = 2^6(-2)^4(1)^8 = 2^{10}$. We conclude

$$a_0 + a_2 + a_4 + \cdots + a_{22} = \frac{0 + 2^{10}}{2} = \boxed{2^9} \quad \text{or} \quad \boxed{512.}$$

□

Problem 5. Suppose m and n are positive integer solutions to the equation

$$mn + 3n + 5m - 16 = 2017.$$

How many possible values for m are there?

Solution. Analogous to completing the square, we can we-write as

$$(m+3)(n+5) - 15 - 16 = 2017, \qquad \text{or} \qquad (m+3)(n+5) = 2048.$$

Since $2048 = 2^{11}$, this implies that both $m+3$ and $n+5$ are both powers of 2 up to 2^{11}, and so if $m+3 = 2^k$, then $n+5 = 2^{11-k}$. This show that (m,n) must be one of the twelve pairs in the following list:

$$(2^{11}-3, 2^0-5), (2^{10}-3, 2^1-5), (2^9-3, 2^2-5), \ldots, (2^2-3, 2^9-5), (2^1-3, 2^{10}-5), (2^0-3, 2^{11}-5)$$

We can further eliminate the first three and last two terms of this list, since they involve either a negative m or n. That leaves $\boxed{7}$ solutions remaining: $m = 2^k - 3$ for $2 \leq k \leq 8$. □

Problem 6. Consider a two-player game that begins with a large number of coins in a pile. On each player's turn, they can remove anywhere from 1 to 2017 coins from the stack. Whoever takes the last coin wins.

If the game begins with 2,020,200 coins in a pile, the first player can guarantee eventual victory with perfect play, but only by playing the correct first move. How many coins should the first player take on the first turn?

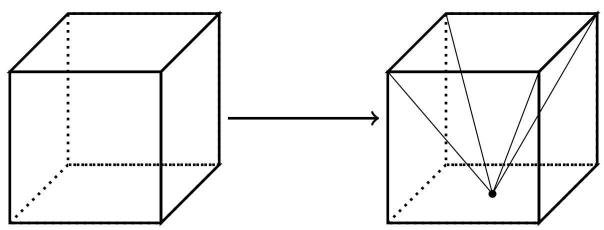
Solution. Player 1 can win by leaving a multiple of 2018 coins in the stack. To see this, if Player 2 begins their turn with a multiple of 2018 coins, then if they take k coins off the pile, Player 1 can respond by taking $2018 - k$, leaving a smaller multiple of 2018. This will continue until Player 1's last move, leaving the smallest possible multiple of 2018, namely 0.

For Player 1 to use the first move to reduce the pile to a multiple of 2018, we need only observe that

$$2020200 = 2018000 + 2018 + 182,$$

so removing $\boxed{182}$ coins will guarantee victory. □

Problem 7. Begin with a solid cube and carve out (remove) a square pyramid whose base is one face of the cube and whose vertex is the center of the opposite face. The resulting solid has smaller volume but larger surface area then the original cube. If the volume decreased by the same amount that the surface area increased, what is the side length of the cube?



Solution. Call the side length of the cube x. The volume of the original cube is x^3 and its surface area is $6x^2$. Since the volume of the pyramid is one third the volume of the cube, carving the figure reduced the volume by $\frac{1}{3}x^3$. For the surface area we will need to compute the area of each triangular face of the removed pyramid. The height of this triangle is the hypotenuse of a right triangle with side lengths x and $x/2$, and so is $\sqrt{\frac{5x^2}{4}}$ by the Pythagorean Theorem. All four triangles in total thus have an area of $4 \cdot \frac{1}{2} \cdot x\sqrt{\frac{5x^2}{4}} = \sqrt{5}x^2$. Thus the carving process has removed an area of x^2 and replaced it with an area of $\sqrt{5}x^2$, a net change of $(\sqrt{5} - 1)x^2$. For this increase to equal the decrease in volume, we need

$$\frac{1}{3}x^3 = (\sqrt{5} - 1)x^2,$$

so $\boxed{x = 3(\sqrt{5} - 1).}$ □

Problem 8. On a trip to the mythical Fields of Mathematics, Artur comes across the infamous three-headed Pemdas of Doom. Each of its three heads, named Manjul, Martin, and Maryam, speak only one sentence per day, and whether they speak the truth or lie is determined solely by the day of the week. Manjul tells the truth only on Monday through Thursday. Martin tells the truth only on Tuesdays, Thursdays, and Saturdays. Maryam tells the truth only on the weekend.

The beast's three heads address Artur as follows:

> **Manjul**: I will tell the truth 700 days from now.
>
> **Martin**: Manjul lied 100 days ago, and will lie 100 days from now as well.
>
> **Maryam**: I will lie 1000 days from now.

What day of the week is it?

Solution. Manjul's statement conveys no information, since the day of the week 700 days from now is the same as the current day of the week, and so his statement amounts to "I am telling the truth," which he could say on any day of the week. So we obtain the solution by analyzing the other two statements:

- Martin must be lying. He claims that there are two days which are 200 days apart on which Manjul will be lying, but 200 days is 28 weeks and 4 days, and there are no two days of the week which are four days apart on which Manjul lies. We conclude that the day is not Tuesday, Thursday, or Saturday.
- Next, consider Maryam's statement. Since 1000 days is 142 weeks and 6 days, the day 1000 days from now is the same as the day it was yesterday. Maryam could only make her claim on a Saturday (truthfully saying that she will lie on a Friday) or a Monday (lying that she will lie on a Sunday).

Combining the two observations, we conclude that it must be a $\boxed{\text{Monday.}}$ □