# MATH FIELD DAY 2017

## Contestants' Instructions
## Team Essay

1. Your team has *forty* minutes to answer this set of questions.

2. All answers must be justified with complete explanations. Your answers should be clear, grammatically correct, and mathematically precise.

3. Your team may turn in at most one answer to each question. Number the answers and submit them in order, starting each problem on a separate page. You are allowed to skip any question and proceed on to later questions. Write on one side of the paper only, and number the pages you turn in consecutively. Do not turn in the problem packets. Write your school name on each page.

4. Your team may be organized in any way you choose. For example, you may designate one team member to write every answer, or you could designate different team members to write different answers. You may also use the chalk board.

5. Only official team members may be in the room during the essay period. Team members may <u>not</u> bring books, calculators, or any other materials into the room.

6. A proctor outside the room will warn you ten minutes before your time is up.

7. Give your answers to the proctor at the end of the period to place in the inner envelope. Please erase all chalk boards in the room and put all scratch paper in the trash can, leaving the room neat.

# Catalan Numbers

A lattice point is a point in the $(x, y)$-plane with integer coordinates. A path is a sequence of moves between lattice points such that each move increases $x$ or $y$ by 1, but not both. A path with 0 moves remains at its starting point.

Throughout this Essay, let $k$ and $m$ be positive integers. The *Catalan number* $s_{k,m}$ is the number of paths from $(0, 0)$ to $(k, mk)$ such that all the lattice points on each path lie on or below the line $y = mx$.

For example, suppose that $k = 2$ and $m = 2$. Figure 1 shows the three paths from $(0, 0)$ to $(2, 4)$ such that all the lattice points on each path lie on or below the line $y = 2x$. Thus, we have $s_{2,2} = 3$.

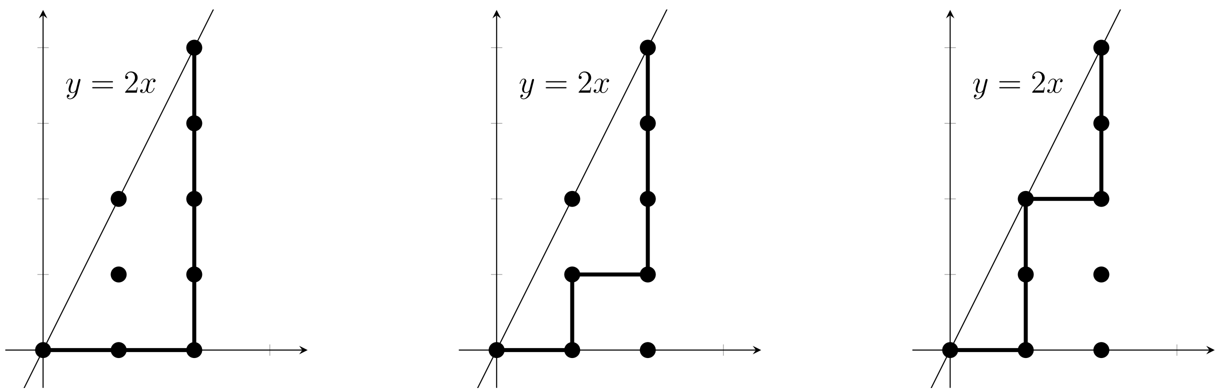


Figure 1

> **Problem 1.** Show that $s_{3,1} = 5$ by drawing the analogue of Figure 1. That is, draw the five paths from $(0, 0)$ to $(3, 3)$ such that all the lattice points on each path lie on or below the line $y = x$. No explanation is needed.

Let $r$ and $n$ be integers with $0 \leq r \leq n$. Let $\binom{n}{r}$ be the number of sets with $r$ elements that can be formed from $n$ objects. For example, the sets of two elements that can be formed from the four letters A, B, C, D are

$$\{A, B\}, \{A, C\}, \{A, D\}, \{B, C\}, \{B, D\}, \{C, D\}$$

and so we have $\binom{4}{2} = 6$.

Note that

$$\binom{n}{0} = 1 \tag{1}$$

for any integer $n \geq 0$, since the empty set is the only set with no elements.

If $1 \leq r \leq n$, the formula

$$\binom{n}{r} = \frac{n(n-1)(n-2)\cdots(n-(r-1))}{r(r-1)(r-2)\cdots 1} \tag{2}$$

gives the value of $\binom{n}{r}$. There are $r$ factors in both the numerator and the denominator. Formula (2) shows, for example, that

$$\binom{4}{2} = \frac{4 \cdot 3}{2 \cdot 1} = 6,$$

as we saw before.

The reason formula (2) holds is this. To form a set of $r$ elements from $n$ objects, we choose the first element in $n$ ways, the second in $n-1$ ways, and so on through the $r$-th element in $n-(r-1)$ ways. These choices give each subset $r(r-1)\cdots 1$ times, since any of the $r$ elements can be chosen first, any of the remaining $r-1$ elements can be chosen second, and so on.

The formula

$$\binom{n}{r} = \frac{n}{r}\binom{n-1}{r-1} \tag{3}$$

holds for $r \geq 1$. In fact, if $r \geq 2$, we have

$$\binom{n}{r} = \frac{n(n-1)(n-2)\cdots(n-(r-1))}{r(r-1)(r-2)\cdots 1} = \frac{n}{r}\left[\frac{(n-1)(n-2)\cdots(n-(r-1))}{(r-1)(r-2)\cdots 1}\right] = \frac{n}{r}\binom{n-1}{r-1},$$

by (2). Formula (3) also holds for $r = 1$, since

$$\binom{n}{1} = \frac{n}{1} = \frac{n}{1}\binom{n-1}{0},$$

by (2) and (1).

This Essay shows that

$$s_{k,m} = \frac{1}{k}\binom{(m+1)k}{k-1} \tag{4}$$

for all positive integers $k$ and $m$. For example, this formula shows that

$$s_{2,2} = \frac{1}{2}\binom{3 \cdot 2}{2-1} = \frac{1}{2}\binom{6}{1} = \frac{1}{2} \cdot \frac{6}{1} = 3$$

and

$$s_{3,1} = \frac{1}{3}\binom{2 \cdot 3}{3-1} = \frac{1}{3}\binom{6}{2} = \frac{1}{3} \cdot \frac{6 \cdot 5}{2 \cdot 1} = 5,$$

as we've already seen directly.

**Problem 2.** Use formula (4) to find the number of paths from $(0,0)$ to $(4,20)$ such that all the lattice points on each path lie on or below the line $y = 5x$.

The first step in deriving formula (4) is to count all the paths between two lattice points.

**Problem 3.** Let $(a,b)$ and $(c,d)$ be lattice points with $a \leq c$ and $b \leq d$. Explain why the total number of paths from $(a,b)$ to $(c,d)$ is

$$\binom{c-a+d-b}{c-a}.$$

**Problem 4.** Use Problem 3 to find the total number of paths from $(0,0)$ to $(2,2)$. Draw each of these paths on a separate grid of lattice points.

The line $y = 2x+1$ lies 1 unit above the line $y = 2x$ in Figure 1. Adding one more vertical edge at the end of each path in Figure 1 gives the three paths from $(0,0)$ to $(2,5)$ that intersect the line $y = 2x+1$ only at the point $(2,5)$ (Figure 2). The next problem generalizes this observation.

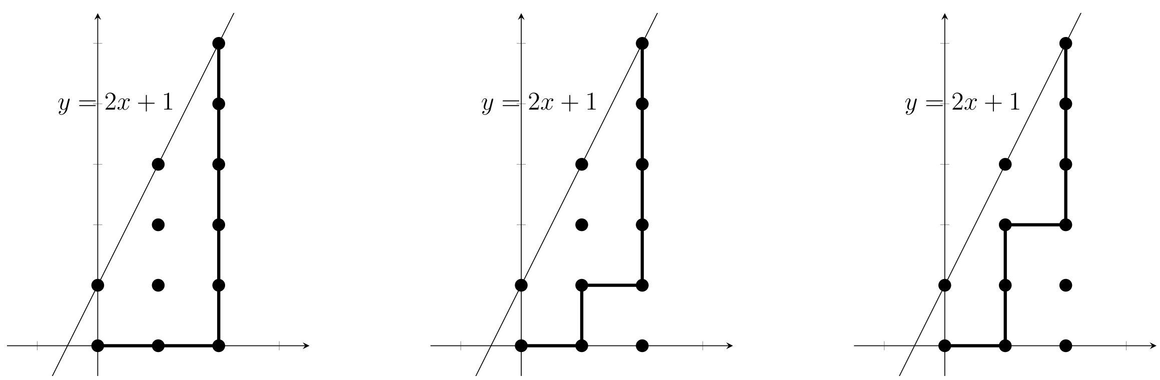


Figure 2

**Problem 5.** Explain why $s_{k,m}$ is the number of paths from $(0,0)$ to $(k, mk+1)$ that intersect the line $y = mx+1$ only at the point $(k, mk+1)$.

We hold the value of $m$ fixed in the remaining problems. To simplify notation, we write $s_{k,m}$ as $s_k$ for $k \geq 1$. In light of Problem 5, it's natural to set

$$s_0 = 1, \tag{5}$$

since the vertical line move from $(0,0)$ to $(0,1)$ is the only path from $(0,0)$ to $(0,1)$, and it intersects the line $y = mx+1$ only at the point $(0,1)$ (Figure 3).

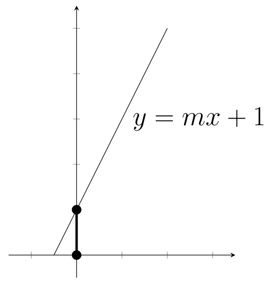


Figure 3

**Problem 6.** Let $i$ be an integer with $0 \le i \le k$. Show that

$$s_i \binom{(m+1)(k-i)}{k-i}$$

is the number of paths from $(0,0)$ to $(k, mk+1)$ that intersect the line $y = mx + 1$ for the first time at the point $(i, mi+1)$. Use Problem 5, equation (5), and Problem 3.

For any integer $n \ge 0$ and numbers $a_0, a_1, \ldots, a_n$, the expression

$$\sum_{i=0}^{n} a_i$$

denotes the sum $a_0 + a_1 + \cdots + a_n$. For example, we have

$$\sum_{i=0}^{3} \frac{12}{i+1} = \frac{12}{1} + \frac{12}{2} + \frac{12}{3} + \frac{12}{4} = 12 + 6 + 4 + 3 = 25.$$

**Problem 7.** Prove that

$$\binom{(m+1)k+1}{k} = \sum_{i=0}^{k} s_i \binom{(m+1)(k-i)}{k-i}. \tag{6}$$

Hint: Use Problem 3 to count the total number of paths from $(0,0)$ to $(k, mk+1)$. Then use Problem 6.

**Problem 8.** Use equations (6), (1), and (3) to show that for $k \ge 1$ we have

$$\left(m + 1 + \frac{1}{k}\right)\binom{(m+1)k}{k-1} = s_k + \sum_{i=0}^{k-1} s_i (m+1) \binom{(m+1)(k-i)-1}{k-i-1}. \tag{7}$$

The next two problems are analogues of Problems 6 and 7 that replace the point $(k, mk+1)$ with the point $(k-1, mk+1)$ one unit to the left.

**Problem 9.** Let $i$ be an integer with $0 \le i \le k-1$. Show that

$$s_i \binom{(m+1)(k-i)-1}{k-i-1}$$

is the number of paths from $(0,0)$ to $(k-1, mk+1)$ that intersect the line $y = mx+1$ for the first time at the point $(i, mi+1)$. Use Problem 5, equation (5), and Problem 3.

**Problem 10.** Prove that

$$\binom{(m+1)k}{k-1} = \sum_{i=0}^{k-1} s_i \binom{(m+1)(k-i)-1}{k-i-1}. \tag{8}$$

Hint: Use Problem 3 to count the total number of paths from $(0,0)$ to $(k-1, mk+1)$. Then use Problem 9.

Problems 8 and 10 give formulas in terms of the Catalan numbers $s_i$ for the total number of paths from $(0,0)$ to $(k, mk+1)$ and $(k-1, mk+1)$. Together these formulas give the value of $s_k$.

**Problem 11.** Conclude from equations (7) and (8) that

$$s_k = \frac{1}{k}\binom{(m+1)k}{k-1}.$$

We've derived the formula

$$s_{k,m} = \frac{1}{k}\binom{(m+1)k}{k-1}$$

for all positive integers $k$ and $m$, as promised. The term "Catalan number' usually means one of the numbers

$$s_{k,1} = \frac{1}{k}\binom{2k}{k-1}$$

for a positive integer $k$. This is the number of paths from $(0,0)$ to $(k,k)$ that don't include points above the diagonal $y = x$. The numbers $s_{k,m}$ for $m \ge 2$ are just one way to generalize the usual Catalan numbers $s_{k,1}$. There is no standard notation for Catalan numbers, and the symbol $s_{k,m}$ designates Catalan number only in this Essay.