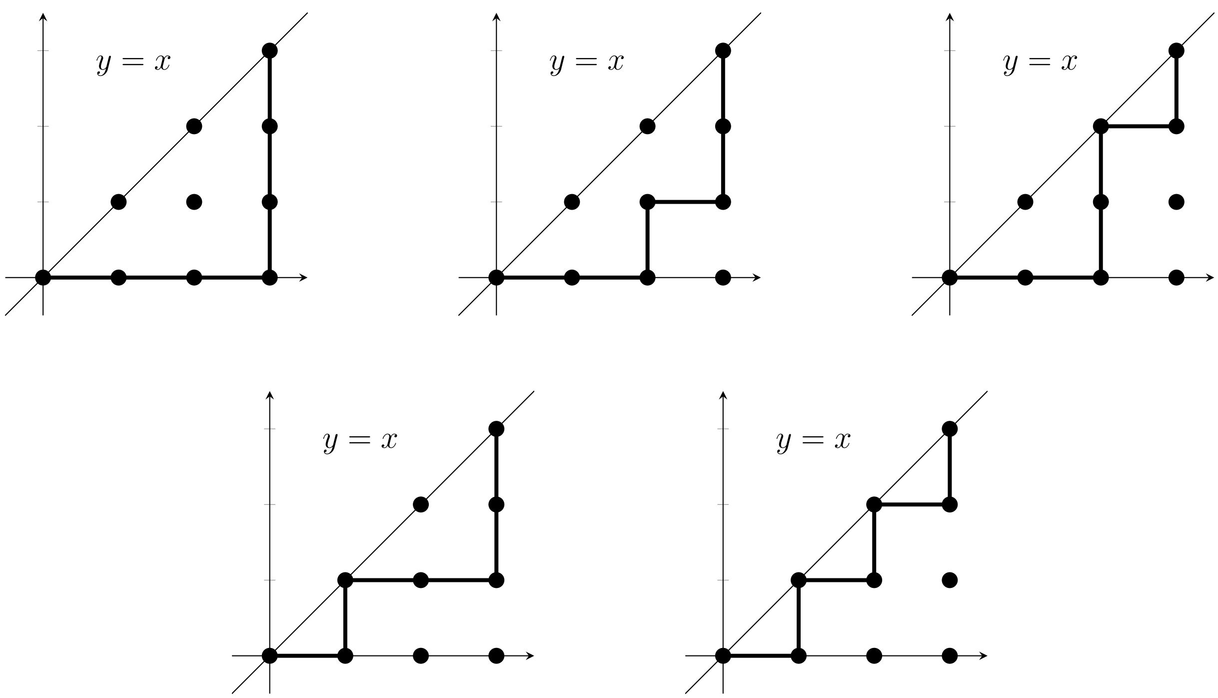
MATH FIELD DAY 2017
Team Essay Solutions

Problem 1. Show that $s_{3,1} = 5$ by drawing the analogue of Figure 1. That is, draw the five paths from $(0,0)$ to $(3,3)$ such that all the lattice points on each path lie on or below the line $y = x$. No explanation is needed.



Solution. □

Problem 2. Use formula (4) to find the number of paths from $(0,0)$ to $(4,20)$ such that all the lattice points on each path lie on or below the line $y = 5x$.

Solution.

$$s_{4,5} = \frac{1}{4}\binom{6 \cdot 4}{4-1} = \frac{1}{4}\binom{24}{3} = \frac{1}{4} \cdot \frac{24 \cdot 23 \cdot 22}{3 \cdot 2 \cdot 1} = 23 \cdot 22 = 506.$$

□

Problem 3. Let (a, b) and (c, d) be lattice points with $a \leq c$ and $b \leq d$. Explain why the total number of paths from (a, b) to (c, d) is

$$\binom{c-a+d-b}{c-a}.$$

Solution. A path from (a, b) to (c, d) moves to the right $c - a$ times and upwards $d - b$ times. It's determined by the sequence of these moves. There are a total of $c-a+d-b$ moves, and the sequence is determined by deciding which $c-a$ of the moves are horizontal (since the remaining moves are vertical). Thus, the number of paths from (a, b) to (c, d) is the number of ways to choose $c - a$ out of $c - a + d - b$ moves to be horizontal, and this number is

$$\binom{c-a+d-b}{c-a}.$$

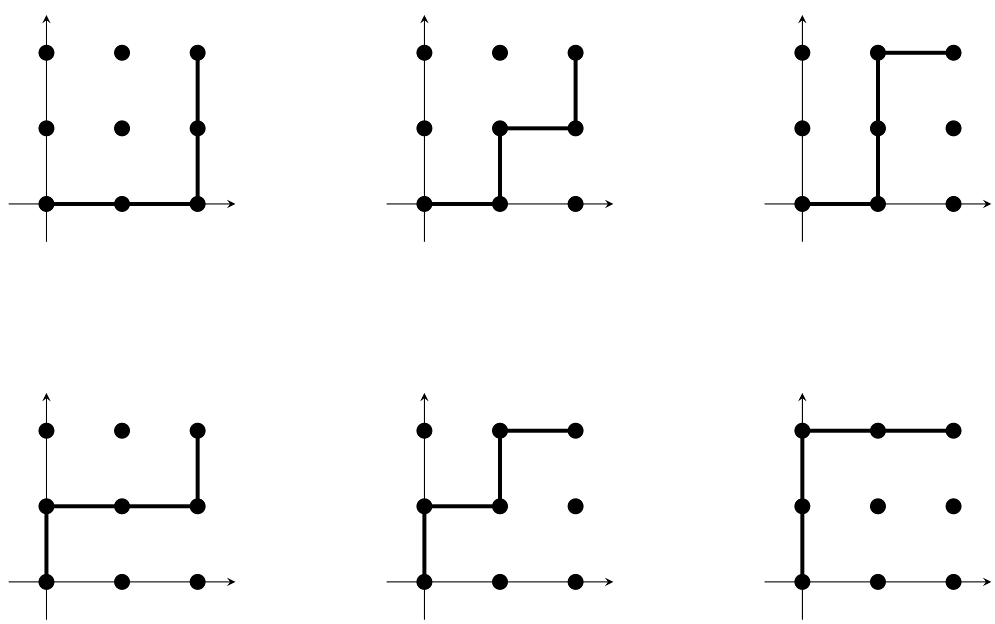
□

Problem 4. Use Problem 3 to find the total number of paths from $(0, 0)$ to $(2, 2)$. Draw each of these paths on a separate grid of lattice points.

Solution.

$$\binom{2-0+2-0}{2-0} = \binom{4}{2} = \frac{4 \cdot 3}{2 \cdot 1} = 6.$$

□



Problem 5. Explain why $s_{k,m}$ is the number of paths from $(0,0)$ to $(k, mk+1)$ that intersect the line $y = mx + 1$ only at the point $(k, mk + 1)$.

Solution. Take any path from $(0,0)$ to (k, mk) whose lattice points all lie on or below the line $y = mx$. Add an edge at the end of the path that moves 1 unit upwards, from (k, mk) to $(k, mk + 1)$. This gives a path from $(0,0)$ to $(k, mk+1)$ that intersects the line $y = mx + 1$ only at $(k, mk + 1)$.

Conversely, consider any path from $(0,0)$ to $(k, mk+1)$ that intersects $y = mx+1$ only at $(k, mk + 1)$. This path ends by moving 1 unit upwards from (k, mk) to $(k, mk + 1)$, and that move is preceded by a path from $(0,0)$ to (k, mk) whose lattice points all lie below $y = mx+1$. These lattice points all lie on or below the line $y = mx$, which lies 1 unit below $y = mx + 1$.

Thus, the number of paths from $(0,0)$ to $(k, mk + 1)$ that intersect $y = mx + 1$ only at $(k, mk+1)$ equals the number of paths from $(0,0)$ to (k, mk) whose lattice points all lie on or below the line $y = mx$. That number is $s_{k,m}$. □

Problem 6. Let i be an integer with $0 \le i \le k$. Show that

$$s_i \binom{(m+1)(k-i)}{k-i}$$

is the number of paths from $(0,0)$ to $(k, mk+1)$ that intersect the line $y = mx+1$ for the first time at the point $(i, mi+1)$. Use Problem 5, equation (5), and Problem 3.

Solution. By Problem 5 and the discussion of Figure 3, s_i is the number of paths from $(0,0)$ to $(i, mi + 1)$ that intersect the line $y = mx + 1$ only at the point $(i, mi + 1)$. By Problem 3, the total number of paths form $(i, mi + 1)$ to $(k, mk + 1)$ is

$$\binom{k-i+(mk+1)-(mi+1)}{k-i} = \binom{k-i+m(k-i)}{k-i} = \binom{(m+1)(k-i)}{k-i}.$$

To get a path from $(0,0)$ to $(k, mk+1)$ that intersects $y = mx+1$ for the first time at $(i, mi+1)$, take a path from $(0,0)$ to $(i, mi + 1)$ that intersects $y = mx + 1$ only at $(i, mi + 1)$ and follow it with any path from $(i, mi + 1)$ to $(k, mk + 1)$. The number of such paths is the product of the numbers s_i and $\binom{(m+1)(k-i)}{k-i}$ in the previous paragraph. □

Problem 7. Prove that

$$\binom{(m+1)k+1}{k} = \sum_{i=0}^{k} s_i \binom{(m+1)(k-i)}{k-i}. \tag{6}$$

Hint: Use Problem 3 to count the total number of paths from $(0,0)$ to $(k, mk+1)$. Then use Problem 6.

Solution. Let p be any path from $(0,0)$ to $(k, mk+1)$. Because the line $y = mx + 1$ lies above $(0,0)$ and contains $(k, mk+1)$, it lies above the path p until their first point of intersection Q. Since p moves only upwards and to the right, it reaches Q after moving upwards. This move is on a line $x = i$ for an integer i from 0 through k (since p moves upwards only on such lines). Then Q is the lattice point $(i, mi+1)$, since Q is on the line $y = mx + 1$.

By Problem 5, the total number of paths from $(0,0)$ to $(k, mk+1)$ is

$$\binom{k-0+mk+1-0}{k-0} = \binom{(m+1)k+1}{k}.$$

By the previous paragraph, this is the sum, as i runs over the integers from 0 through k, of the number of paths from $(0,0)$ to $(k, mk+1)$ that intersect $y = mx+1$ first at $i(mi+1)$. Together with Problem 6, the last two sentences show that

$$\binom{(m+1)k+1}{k} = \sum_{i=0}^{k} s_i \binom{(m+1)(k-i)}{k-i}.$$

□

Problem 8. Use equations (6), (1), and (3) to show that for $k \geq 1$ we have

$$\left(m+1+\frac{1}{k}\right)\binom{(m+1)k}{k-1} = s_k + \sum_{i=0}^{k-1} s_i (m+1) \binom{(m+1)(k-i)-1}{k-i-1}. \tag{7}$$

Solution. Since $k \geq 1$, equation (3) shows that

$$\binom{(m+1)k+1}{k} = \frac{(m+1)k+1}{k}\binom{(m+1)k}{k-1} = \left(m+1+\frac{1}{k}\right)\binom{(m+1)k}{k-1}.$$

Thus, the left sides of equations (6) and (7) are equal.

The right side of (6) is the sum as i runs over the integers from 0 through k of the terms

$$s_i \binom{(m+1)(k-i)}{k-i}. \tag{$*$}$$

For $i = k$, this term is $s_k\binom{0}{0} = s_k$, since $\binom{0}{0} = 1$ by (1). Thus, the right side of (6) is s_k plus the sum of the terms $(*)$ for all integers i from 0 through $k-1$. For each such term, we have $k - i \geq 1$ (since $i \leq k-1$), and so equation (3) shows that $(*)$ equals

$$s_i \frac{(m+1)(k-i)}{k-i}\binom{(m+1)(k-i)-1}{k-i-1} = s_i(m+1)\binom{(m+1)(k-1)-1}{k-i-1}.$$

Accordingly, the right side of (6) is

$$s_k + \sum_{i=0}^{k-1} s_i(m+1)\binom{(m+1)(k-i)-1}{k-i-1},$$

which is the right side of (7). Thus, corresponding sides of (6) and (7) are equal. Since (6) holds (by Problem 7), so does (7). □

Problem 9. Let i be an integer with $0 \leq i \leq k-1$. Show that

$$s_i\binom{(m+1)(k-i)-1}{k-i-1}$$

is the number of paths from $(0,0)$ to $(k-1, mk+1)$ that intersect the line $y = mx+1$ for the first time at the point $(i, mi+1)$. Use Problem 5, equation (5), and Problem 3.

Solution. By Problem 5 and the discussion of Figure 3, s_i is the number of paths from $(0,0)$ to $(i, mi+1)$ that intersect the line $y = mx+1$ only at the point $(i, mi+1)$. By Problem 3, the total number of paths from $(i, mi+1)$ to $(k-1, mk+1)$ is

$$\binom{(k-i)-i+(mk+1)-(mi+1)}{k-1-i} = \binom{k-i+m(k-i)-1}{k-i-1} = \binom{(m+1)(k-i)-1}{k-i-1}.$$

To get a path from $(0,0)$ to $(k-1, mk+1)$ that intersects $y = mx+1$ for the first time at $(i, mi+1)$, take a path from $(0,0)$ to $(i, mi+1)$ that intersects $y = mx+1$ only at $(i, mi+1)$ and follow it with any path from $(i, mi+1)$ to $(k-1, mk+1)$. The number of possibilities is the product of the numbers s_i and $\binom{(m+1)(k-i)-1}{k-i-1}$ in the previous paragraph. □

Problem 10. Prove that

$$\binom{(m+1)k}{k-1} = \sum_{i=0}^{k-1} s_i\binom{(m+1)(k-i)-1}{k-i-1}. \tag{8}$$

Hint: Use Problem 3 to count the total number of paths from $(0,0)$ to $(k-1, mk+1)$. Then use Problem 9.

Solution. Let p be any path from $(0,0)$ to $(k-1, mk+1)$. The line $y = mx+1$ is above $(0,0)$ and below $(k-1, mk+1)$ (which is 1 unit to the left of $(k, mk+1)$). Thus $y = mx+1$ lies above the path p until their first point of intersection Q. Because p moves only upwards and to the right, it reaches Q after moving upwards. This move occurs on a line $x = i$ for an integer i from 0 through $k-1$. Then Q is the lattice point $(i, mi+1)$, since Q lies on the line $y = mx+1$.

By Problem 3, the total number of paths from $(0,0)$ to $(k-1, mk+1)$ is

$$\binom{(k-1) - 0 + (mk+1) - 0}{k-1-0} = \binom{(m+1)k}{k-1}.$$

By the previous paragraph, this is the sum, as i runs over the integers from 0 through $k-1$, of the number of paths from $(0,0)$ to $(k-1, mk+1)$ that intersects $y = mx+1$ first at $(i, mi+1)$. Together with Problem 9, the last two sentences show that

$$\binom{(m+1)k}{k-1} = \sum_{i=0}^{k-1} s_i \binom{(m+1)(k-i)-1}{k-i-1}.$$

□

Problem 11. Conclude from equations (7) and (8) that

$$s_k = \frac{1}{k}\binom{(m+1)k}{k-1}.$$

Solution. Using the distributive law to factor $m+1$ out of every factor in the summation on the right side of (7) gives

$$\left(m+1+\frac{1}{k}\right)\binom{(m+1)k}{k-1} = s_k + (m+1)\sum_{i=0}^{k-1} s_i \binom{(m+1)(k-i)-1}{k-i-1}.$$

Using (8) to substitute for the summation on the right gives

$$\left(m+1+\frac{1}{k}\right)\binom{(m+1)k}{k-1} = s_k + (m+1)\binom{(m+1)k}{k-1}.$$

Solving for s_k gives

$$s_k = \left(m+1+\frac{1}{k}\right)\binom{(m+1)k}{k-1} - (m+1)\binom{(m+1)k}{k-1} = \frac{1}{k}\binom{(m+1)k}{k-1}.$$

□