2018 MATH FIELD DAY
LEAP FROG PROBLEMS

Problem 1. A sequence is defined by the following rule: $a_1 = 2$, and for each $n \geq 2$, we have

$$a_n = \frac{1}{1 - a_{n-1}}.$$

Compute $a_1 + a_2 + a_3 + \cdots + a_{2018}$.

Problem 2. For a real number m, the graphs of the functions

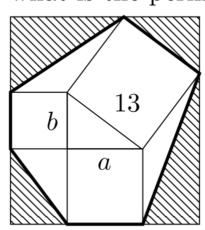
$$f(x) = x^2 + (m+3)x + (m+4)$$

and

$$g(x) = x^2 + (m+7)x + (2m+9)$$

are parabolas. For which value of m are the vertices of these two parabolas the closest to each other?

Problem 3. In the following diagram, squares are constructed on the sides of a right triangle with hypotenuse 13. The squares are connected as shown to construct a hexagon and finally, the hexagon is inscribed in a rectangle. If the area of the of the region inside that rectangle but outside the hexagon (the shaded region in the image) has an area of 84 square units, what is the perimeter of the initial right triangle?



Problem 4. Consider the prime factorizations of two integers m and n:

$$m = 2^5 \cdot 3^4 \cdot 5^3 \cdot 7^2 \cdot 11$$

$$n = 2^a \cdot 3^b \cdot 5^c \cdot 7^d \cdot 11^e$$

Suppose that if you pick a random divisor of m and a random divisor of n, their sum is even with probability $\frac{1}{2}$. Find a.

Problem 5. A two-player game begins with 2018 chips in a pile, and on each player's turn they take a number of chips from the pile. They must take at least one chip and can take at most half of the chips remaining in the stack (rounded up). The winner is the person who takes the last chip. How many chips should the first player take on the first turn in order to guarantee eventual victory, assuming best play?

Problem 6. Find the number of 6-digit numbers n there are such that:

- Each digit of n is a 0, 1, or 7.
- None of those three digits appear exactly once in n.
- n is divisible by 6.

(The first digit of a number cannot be zero).

Problem 7. Find both angles θ (in radians) with $0 \leq \theta \leq \frac{\pi}{2}$ which satisfy the equation

$$4^{1-\frac{\cos(4\theta)}{2}} = 8^{1-\sin(2\theta)}.$$

Problem 8. For positive real numbers x and y, consider the following list of 6 numbers:

$$x\,,\,y\,,\,0\,,\,8\,,\,1\,,\,3$$

Find the area of the set of points (x, y) in the first quadrant for which the median of these 6 numbers is at least as large as their mean.