2018 MATH FIELD DAY
LEAP FROG PROBLEMS AND SOLUTIONS

Problem 1. A sequence is defined by the following rule: $a_1 = 2$, and for each $n \geq 2$, we have

$$a_n = \frac{1}{1 - a_{n-1}}.$$

Compute $a_1 + a_2 + a_3 + \cdots + a_{2018}$.

Solution. The first four terms are $a_1 = 2$, $a_2 = \frac{1}{1-2} = -1$, $a_3 = \frac{1}{1-(-1)} = \frac{1}{2}$, and $a_4 = \frac{1}{1-\frac{1}{2}} = 2$. Since $a_4 = a_1$, the pattern begins to repeat, i.e., we will also have $a_5 = a_2$, $a_6 = a_3$, etc. The sum of each set of three consecutive terms is thus the same as $a_1 + a_2 + a_3 = 2 - 1 + \frac{1}{2} = \frac{3}{2}$. So

$$\begin{aligned} a_1 + a_2 + a_3 + \cdots + a_{2018} &= (a_1 + a_2 + a_3) + (a_4 + a_5 + a_6) + \cdots + (a_{2014} + a_{2015} + a_{2016}) + a_{2017} + a_{2018} \\ &= 672 \cdot \frac{3}{2} + 2 - 1 = \boxed{1009.} \end{aligned}$$

□

Problem 2. For a real number m, the graphs of the functions

$$f(x) = x^2 + (m+3)x + (m+4)$$

and

$$g(x) = x^2 + (m+7)x + (2m+9)$$

are parabolas. For which value of m are the vertices of these two parabolas the closest to each other?

Solution. For an arbitrary quadratic $ax^2 + bx + c$, the vertex occurs at $\left(\frac{-b}{2a}, c - \frac{b^2}{4a}\right)$. So the vertices of the graph of f occur respectively at

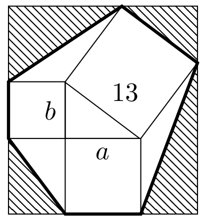
$$\left(-\frac{m+3}{2},\ m+4 - \frac{(m+3)^2}{2}\right) \qquad \text{and} \qquad \left(-\frac{m+7}{2},\ 2m+9 - \frac{(m+7)^2}{2})\right)$$

The distance formula gives the distance between these two points as

$$\begin{aligned} d &= \sqrt{\left(\frac{m+7}{2} - \frac{m+3}{2}\right)^2 + \left(2m+9 - (m+4) + \frac{(m+3)^2}{2} - \frac{(m+7)^2}{2}\right)^2} \\ &= \sqrt{(2)^2 + (m+5-2m-10)^2} \\ &= \sqrt{m^2 + 10m + 29} \\ &= \sqrt{(m+5)^2 + 2} \end{aligned}$$

which has its smallest possible value when $\boxed{m = -5.}$ □

Problem 3. In the following diagram, squares are constructed on the sides of a right triangle with hypotenuse 13. The squares are connected as shown to construct a hexagon and finally, the hexagon is inscribed in a rectangle. If the area of the of the region inside that rectangle but outside the hexagon (the shaded region in the image) has an area of 84 square units, what is the perimeter of the initial right triangle?



Proof. Note by the Pythagorean Theorem that $a^2 + b^2 = 13^2$. Now, using each of the edges of the 13-by-13 square as a hypotenuse of a congruent to the a-b-13 right triangle, we see that the exterior rectangle is a $(2a + b)$-by-$(2b + a)$ rectangle, and so has area

$$(2a + b)(2b + a) = 4ab + 2b^2 + 2a^2 + ab = 2(a^2 + b^2) + 5ab = 2 \cdot 13^2 + 5ab.$$

On the other hand, the area inside the hexagon consists of three squares and four triangles of area $\frac{1}{2}ab$ for a total area of

$$a^2 + b^2 + 13^2 + 4\left(\frac{1}{2}ab\right) = 2 \cdot 13^2 + 2ab.$$

The difference between these two areas, $3ab$, is given to us to be 84, so $ab = 28$. Now we compute

$$(a + b)^2 = a^2 + b^2 + 2ab = 13^2 + 2(28) = 169 + 56 = 225,$$

so $a + b = 15$ and the perimeter of the right triangle is $a + b + 13 = \boxed{28.}$ □

Problem 4. Consider the prime factorizations of two integers m and n:

$$m = 2^5 \cdot 3^4 \cdot 5^3 \cdot 7^2 \cdot 11$$
$$n = 2^a \cdot 3^b \cdot 5^c \cdot 7^d \cdot 11^e$$

Suppose that if you pick a random divisor of m and a random divisor of n, their sum is even with probability $\frac{1}{2}$. Find a.

Solution. Let p be the probability that the chosen divisor of m is odd, and q the probability that the chosen divisor of n is odd. Then the probability that their sum is even is $pq + (1-p)(1-q)$ (it must be that either both divisors are even or both divisors are odd). For this sum to equal $\frac{1}{2}$, we need $pq + (1-p)(1-q) = \frac{1}{2}$, or equivalently, $(2p - 1)(2q - 1) = 0$. Since $p \neq \frac{1}{2}$ (only $\frac{1}{6}$ of m's divisors are odd), we thus need to have $q = \frac{1}{2}$. Likewise, since $\frac{1}{a+1}$ of n's divisors are odd, setting $\frac{1}{a+1} = \frac{1}{2}$ shows that this is achieved if and only if $\boxed{a = 1.}$ □

Problem 5. A two-player game begins with 2018 chips in a pile, and on each player's turn they take a number of chips from the pile. They must take at least one chip and can take at most half of the chips remaining in the stack (rounded up). The winner is the person who takes the last chip. How many chips should the first player take on the first turn in order to guarantee eventual victory, assuming best play?

Proof. Think of the game as a function of the number of starting chips. It is clear that it is a win for the 1st player if the game starts with 1 chip, and a loss for the first player if it starts with 2 chips. Starting with $n = 3, 4,$ or 5 chips is back to being a win for the first player as by taking 1, 2, or 3 chips we can leave the second player in the losing position of 2 chips.

The pattern becomes clear – 6 is losing for player 1, and then 7 through 13 must be winning since we can reduce the pile from any of these number to 6. In general, given any losing number n, the next losing number is $2n + 2$. The sequence starts 2, 6, 14, and we note that the rule $n \to 2n + 2$ preserves the property of being two less than a power of 2. Thus the losing positions are those numbers of the form $2^k - 2$. If, therefore, we start with 2018 chips, the next closest losing position is $1024 - 2 = 1022$ chips, so the first player should take $2018 - 1022 = \boxed{996}$ chips on their first turn. □

Problem 6. Find the number of 6-digit numbers n there are such that:

- Each digit of n is a 0, 1, or 7.
- None of those three digits appear exactly once in n.
- n is divisible by 6.

(The first digit of a number cannot be zero).

Solution. To be divisible by 6 is equivalent to being divisible by 2 and by 3. So since n must be even, its last digit must be a 0. Since n is divisible by 3, so the sum of its digits must be a multiple of 3. The only way this can happen is if n contain three 0's, and three non-zero digits. The 3 non-zero digits can't be a 1 and two 7's or a 7 and two 1's, by the second rule. So n must consist of either three 0's and three 1's, or three 0's and three 7's, in some order. Since the first digit *can't* be a 0, and the last digit *must* be a 0, we have only the following possibilities using three 0's and three 1's:

111000 101100 100110 101010 110100 110010

These 6, along with the other 7 obtained by replacing all of the 1's with 7's, give a total of $\boxed{12}$ such numbers. □

Problem 7. Find both angles θ (in radians) with $0 \le \theta \le \frac{\pi}{2}$ which satisfy the equation

$$4^{1-\frac{\cos(4\theta)}{2}} = 8^{1-\sin(2\theta)}.$$

Solution. Among other options, we can re-write both sides using a base of 2:

$$2^{2-\cos(4\theta)} = 2^{3-3\sin(2\theta)}$$

Dividing by 2, using the identity $1-\cos(4\theta) = 2\sin^2(2\theta)$ and taking the base-2 logarithm of both sides leaves us with

$$2\sin^2(2\theta) = 2 - 3\sin(2\theta),$$

or

$$0 = 2\sin^2(2\theta) + 3\sin(2\theta) - 2 = (2\sin(2\theta)-1)(\sin(2\theta)+2)$$

which implies that $\sin(2\theta) = \frac{1}{2}$ (since it can't be -2), and so we finally conclude that $\theta = \boxed{\frac{\pi}{12}, \frac{5\pi}{12}}$ □

Problem 8. For positive real numbers x and y, consider the following list of 6 numbers:

$$x\,,\,y\,,\,0\,,\,8\,,\,1\,,\,3$$

Find the area of the set of points (x, y) in the first quadrant for which the median of these 6 numbers is at least as large as their mean.

Solution. Let us write M for the median and m for the mean, and suppose x and y are values for which $M \ge m$. Then $m = \frac{12+x+y}{6} = 2 + \frac{x+y}{6}$ regardless of x and y, whereas M is the midpoint of the third-largest and fourth-largest values in the set. We first show that for $M \ge 2 + \frac{x+y}{6}$ to hold, it must be that 0, 1, and 3 must be the three smallest numbers in the list, so that M is the average of 3 and the fourth smallest.

First, to have $M \ge 2 + \frac{x+y}{6}$ we must certainly have $x, y > 1$, else the median would be less than 1. Since 0 and 1 are thus the smallest two numbers of the six, we conclude that M is the average of the two smallest of 3, 8, x, and y, and so $M \le \frac{x+y}{2}$. Combining, we learn that

$$\frac{x+y}{2} \ge M \ge 2 + \frac{x+y}{6},$$

which implies that $x + y \ge 6$, and so $M \ge 2 + \frac{x+y}{6} \ge 3$. Finally, that $M \ge 3$ in turn forces $x \ge 3$ and $y \ge 3$, so 3 must be the fourth largest number in the set, and M is the average and the smallest of x, y, or 8.

Returning to the calculation, the above shows that M is the smallest of $\frac{3+x}{2}$, $\frac{3+y}{2}$, and $\frac{3+8}{2}$, and so the condition $M \ge 2 + \frac{x+y}{6}$ holds if and only if all of the following three inequalities hold:

$$\frac{3+x}{2} \ge 2 + \frac{x+y}{6}$$
$$\frac{3+y}{2} \ge 2 + \frac{x+y}{6}$$
$$\frac{3+8}{2} \ge 2 + \frac{x+y}{6}$$

which can be re-written respectively as

$$y \le 2x - 3$$
$$x \le 2y - 3$$
$$x + y \le 21$$

The solution set to these three inequalities is the triangular region with vertices at $(3, 3)$, $(8, 13)$, and $(13, 8)$. The area of this triangle is the area of the 10×10 square with vertices at $(3, 3)$, $(13, 3)$, $(3, 13)$, and $(13, 13)$ minus the areas of three extraneous triangles of respective areas of 25, 25, and 12.5. We get a final answer of $100 - 25 - 25 - 12.5 = \boxed{37.5.}$ □