2018 Mad Hatter A

1. The system with unknowns x and y

$$\begin{aligned} ax + 2y &= 5 \\ 2x - by &= 10 \end{aligned}$$

has infinitely many solutions. Find the value of $a + b$.

2. If $\sqrt{n^3 + n^3 + n^3 + n^3 + n^3} = 25$, what is n?

3. The equation

$$(\sqrt{2})^x = x$$

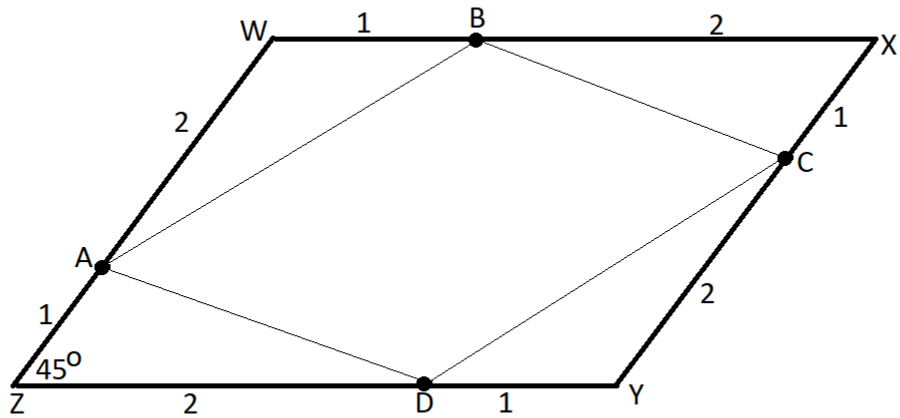
has a solution $x = 2$. It has one more positive solution. What is it?

4. In how many ways can you walk up a stairway that has 7 steps if you can take 1 or 2 steps at a time?

5. Suppose that $k!$ is a multiple of 5^{10}. What is the smallest possible positive integer k?

6. Let k be a real number. Suppose that, for all x, $f(x) = kx + 3$. If $f^{-1}(15) = 2$, find k.

7. Rhombus $WXYZ$ has sides of length 3, and contains a 45^o angle, as shown. Find the area of parallelogram $ABCD$.

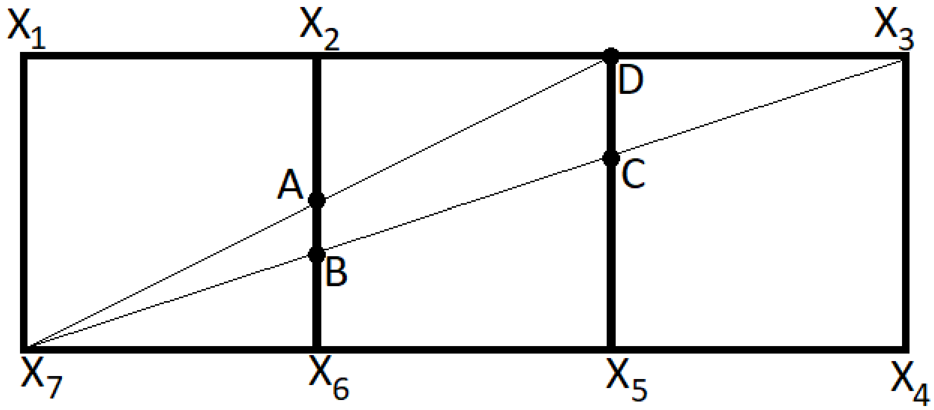


8. How many real solutions does the equation

$$x^2 + x - 3 = |x|$$

have?

9. Each of the quadrilaterals $X_1X_2X_6X_7$, $X_2DX_5X_6$, and $DX_3X_4X_5$ is a square of side length 1. Find the area of quadrilateral $ABCD$.



10. What positive integer n satisfies $\log_{10}(225!) - \log_{10}(223!) = 1 + \log_{10}(n!)$?

11. For what value of k is the circle $x^2 + y^2 = k$ tangent to the line $x + 2y = 5$?

12. Find the product of all real values of p that satisfy the equation $6|p - 6| = |p + 6|$.

13. If $\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{4^2}\right)\cdots\left(1 - \frac{1}{2018^2}\right) = \frac{2019}{2018}a$, what is a?

14. The number 2 is a double root of the equation $x^3 + x^2 + ax = b$. What is b?

15. Find the area of the circle that passes through the points $(0, 0)$, $(1, 1)$ and $(3, -3)$.

16. Find the shortest distance between a point on the curve $x^2 + y^2 = 9$ and a point on the curve $(x - 8)^2 + (y - 6)^2 = 4$.

17. Suppose $\theta = \pi/32$. Evaluate $\sin(\theta)\cos(\theta)\cos(2\theta)\cos(4\theta)\cos(8\theta)$.

18. Let $i = \sqrt{-1}$. Find the sum of

$$\frac{1}{i} + \frac{3}{i^3} + \frac{5}{i^5} + \frac{7}{i^7} + \cdots + \frac{53}{i^{53}}$$

19. The set of numbers $\{-3, 1, 3, a, b, 7, 10\}$ has mean 5 and median 4. Evaluate $(a - b)^2$.

20. You are given that $xy = 6$ and $x^2y + xy^2 + x + y = 63$. Calculate the value of $x^2 + y^2$.

21. Of the numbers $1, 2, 3, \ldots, 20$, which are the sum of 3 consecutive integers? List them.

22. The integers x, $3x + 1$, and $6x + 2$ form a geometric sequence. Find the value of x.

23. How many real solutions does the equation $x^5 - x^4 + x^3 - x^2 + x - 1 = 0$ have?

24. Evaluate $\sqrt[\log_2(2018)]{2018}$.

25. Evaluate $(\sqrt[3]{2} - 1)(1 + \sqrt[3]{2} + \sqrt[3]{4} + \sqrt[3]{8} + \sqrt[3]{16} + \cdots + \sqrt[3]{256})$.

26. What is the y-coordinate of the center of the circle that passes through (-1,2), (3,2), and (5,4)?

27. Suppose that c is a nonzero real number and the real part of each solution of the equation

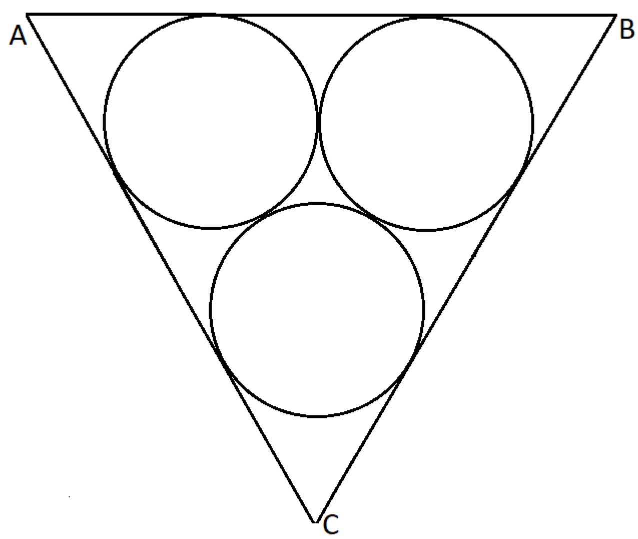
$$\frac{1}{x+c} = \frac{1}{x} + \frac{1}{c}$$

is 1. What is c?

28. If 5 kids can build 5 sets of Lego Star Wars Millennium Falcon in 5 days, how many days will it take 10 kids to build 10 sets of Lego Star Wars Millennium Falcon?

29. If $0° < x < 90°$ and $\sin(2x) = 0.21$. Evaluate $\sin x + \cos x$.

30. Each of the circles has radius 1 and is tangent to the other two circles and to two sides of triangle ABC. What is the length of each side of the triangle?



31. A box contains a two-headed coin, a two-tailed coin, and an ordinary fair coin. You choose one of the coins at random, and flip it twice. What is the probability that the coin flips result in two heads?

32. Consider the statements:

" No grack is a pling."
" At least one grack is a blodget."

Which of the following statements follow logically from these premises? Select all correct answers.

(a) At least one blodget is not a pling.

(b) At least one pling is not a blodget.
(c) All plings are blodgets.
(d) No pling is a blodget.

33. The length, width, and height of a rectangular box are $3a$ inches, $2a$ inches, and a inches respectively, where a is a positive number. If the volume of the box is 48 cubic inches, how long is its diagonal in inches?

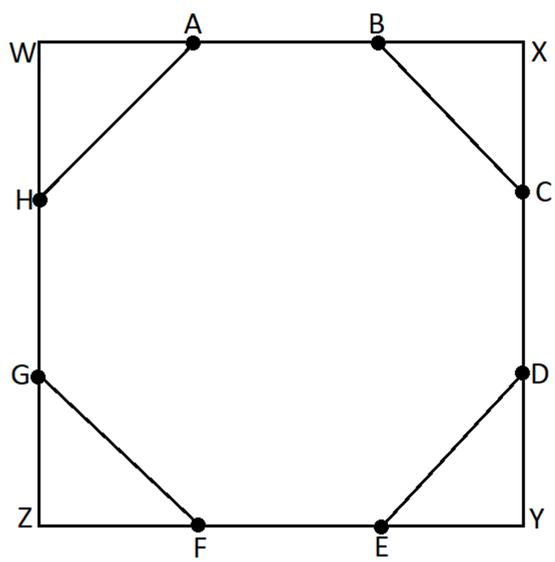
34. Let $f(x) = (x^5 - 1) \cdot (x^3 + 1)$ and $g(x) = (x^2 - 1) \cdot (x^2 - x + 1)$. If $h(x)$ is a polynomial such that $f(x) = g(x) \cdot h(x)$ for all x, what is the value of $h(1)$?

35. A regular polygon of n sides inscribed in a circle of radius r has area $3r^2$. What is the value of n?

36. The coefficient of x^5 in the polynomial $(2x + b)^8$ is -14336. What is b?

37. Suppose that a and b are real numbers and $5^a = 3$ and $3^b = 125$. Find the value of ab.

38. Square $WXYZ$ has sides of length 1. Each of the vertices of regular octagon $ABCDEFGH$ lies on a side of the square. Find the length of each side of the regular octagon.



39. How many numbers from 1 to 1000 contain exactly two 5's among its digits?

40. How many positive integers less than 20,000 are both perfect squares and perfect cubes?

2018 Mad Hatter B

1. The system with unknowns x and y

$$\begin{aligned} ax + 2y &= 5 \\ 2x - by &= 10 \end{aligned}$$

has infinitely many solutions. Find the value of $a + b$.

2. If $\sqrt{n^3 + n^3 + n^3 + n^3 + n^3} = 25$, what is n?

3. The equation

$$(\sqrt{2})^x = x$$

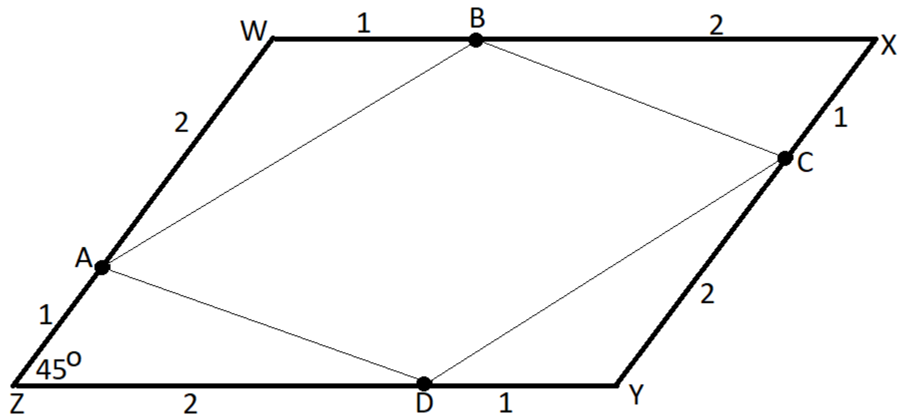
has a solution $x = 2$. It has one more positive solution. What is it?

4. In how many ways can you walk up a stairway that has 7 steps if you can take 1 or 2 steps at a time?

5. Suppose that $k!$ is a multiple of 5^{10}. What is the smallest possible positive integer k?

6. Let k be a real number. Suppose that, for all x, $f(x) = kx + 3$. If $f^{-1}(15) = 2$, find k.

7. Rhombus $WXYZ$ has sides of length 3, and contains a 45^o angle, as shown. Find the area of parallelogram $ABCD$.

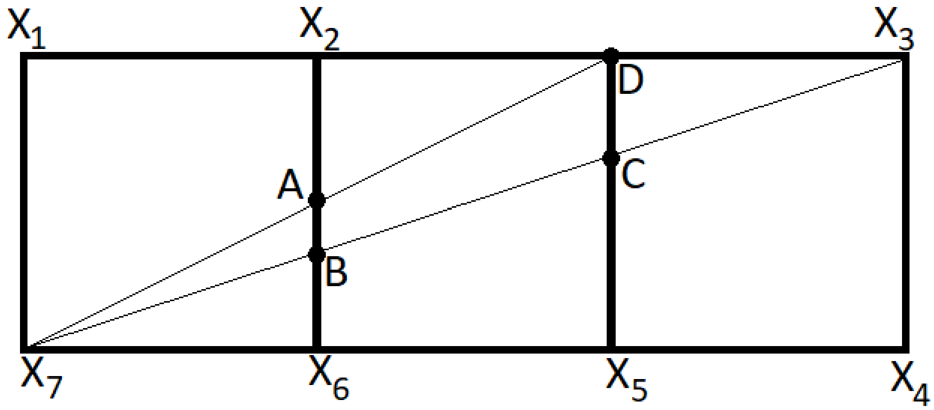


8. How many real solutions does the equation

$$x^2 + x - 3 = |x|$$

have?

9. Each of the quadrilaterals $X_1X_2X_6X_7$, $X_2DX_5X_6$, and $DX_3X_4X_5$ is a square of side length 1. Find the area of quadrilateral $ABCD$.



10. Two rectangles with integer sides have diagonals of the same length, $\sqrt{85}$, but different areas. Find the sum of their areas.

11. For what value of k is the circle $x^2 + y^2 = k$ tangent to the line $x + 2y = 5$?

12. Find the product of all real values of p that satisfy the equation $6|p - 6| = |p + 6|$.

13. If $\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{4^2}\right)\cdots\left(1 - \frac{1}{2018^2}\right) = \frac{2019}{2018}a$, what is a?

14. The number 2 is a double root of the equation $x^3 + x^2 + ax = b$. What is b?

15. Find the area of the circle that passes through the points $(0, 0)$, $(1, 1)$ and $(3, -3)$.

16. Find the shortest distance between a point on the curve $x^2 + y^2 = 9$ and a point on the curve $(x - 8)^2 + (y - 6)^2 = 4$.

17. When the number written xx in decimal notation is multiplied by 99, the digit in the 100's place is 5. What is x?

18. Let $i = \sqrt{-1}$. Find the sum of

$$\frac{1}{i} + \frac{3}{i^3} + \frac{5}{i^5} + \frac{7}{i^7} + \cdots + \frac{53}{i^{53}}$$

19. The set of numbers $\{-3, 1, 3, a, b, 7, 10\}$ has mean 5 and median 4. Evaluate $(a - b)^2$.

20. You are given that $xy = 6$ and $x^2y + xy^2 + x + y = 63$. Calculate the value of $x^2 + y^2$.

21. Of the numbers $1, 2, 3, \ldots, 20$, which are the sum of 3 consecutive integers? List them.

22. The integers x, $3x + 1$, and $6x + 2$ form a geometric sequence. Find the value of x.

23. How many real solutions does the equation $\quad x^5 - x^4 + x^3 - x^2 + x - 1 = 0 \quad$ have?

24. Find the area of a rhombus with sides of length 6 and one diagonal of length 4.

25. Evaluate $(\sqrt[3]{2}-1)(1+\sqrt[3]{2}+\sqrt[3]{4}+\sqrt[3]{8}+\sqrt[3]{16}+\cdots+\sqrt[3]{256})$.

26. What is the y-coordinate of the center of the circle that passes through (-1,2), (3,2), and (5,4)?

27. Suppose that c is a nonzero real number and the real part of each solution of the equation

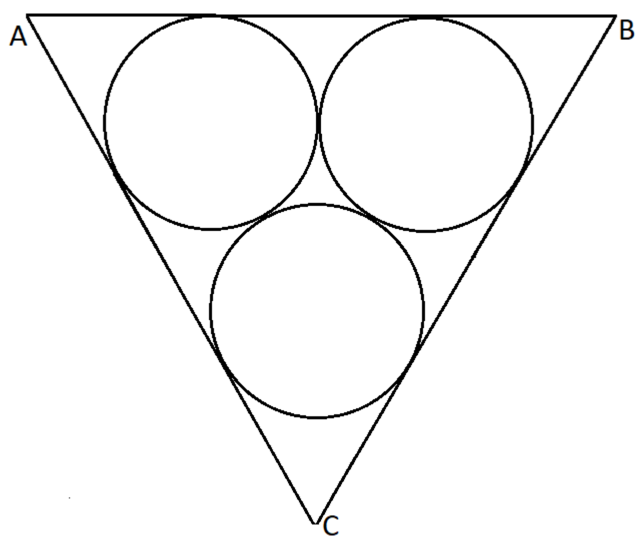
$$\frac{1}{x+c}=\frac{1}{x}+\frac{1}{c}$$

is 1. What is c?

28. If 5 kids can build 5 sets of Lego Star Wars Millennium Falcon in 5 days, how many days will it take 10 kids to build 10 sets of Lego Star Wars Millennium Falcon?

29. If the volume of a sphere is x in^3 and its surface area is x in^2. What is its diameter in inches?

30. Each of the circles has radius 1 and is tangent to the other two circles and to two sides of triangle ABC. What is the length of each side of the triangle?



31. A box contains a two-headed coin, a two-tailed coin, and an ordinary fair coin. You choose one of the coins at random, and flip it twice. What is the probability that the coin flips result in two heads?

32. Person A lies on Mondays, Tuesdays, and Wednesdays, and tells the truth on the other days of the week. Person B lies on Thursdays, Fridays, and Saturdays and tells the truth on the other days of the week. One day they both say, “Yesterday was one of my lying days.” On what day do they say this?

33. The length, width, and height of a rectangular box are $3a$ inches, $2a$ inches, and a inches

respectively, where a is a positive number. If the volume of the box is 48 cubic inches, how long is its diagonal in inches?

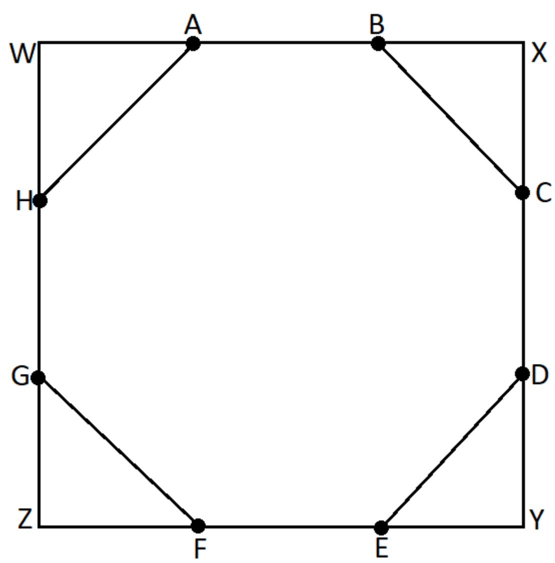
34. Let $f(x) = (x^5 - 1) \cdot (x^3 + 1)$ and $g(x) = (x^2 - 1) \cdot (x^2 - x + 1)$. If $h(x)$ is a polynomial such that $f(x) = g(x) \cdot h(x)$ for all x, what is the value of $h(1)$?

35. Find the area of the region determined by $|x| + |y| \geq 2$ and $x^2 + y^2 \leq 4$.

36. The coefficient of x^5 in the polynomial $(2x + b)^8$ is -14336. What is b?

37. A student was supposed to solve a set of mathematics problems at a rate of 20 problems per day. He solved half of the problem set at a rate of 10 problems per day. To make up for lost time, he solved the remaining half of the problems at a rate of 30 problems per day. When he was done, on average how many problems had he solved per day?

38. Square $WXYZ$ has sides of length 1. Each of the vertices of regular octagon $ABCDEFGH$ lies on a side of the square. Find the length of each side of the regular octagon.



39. How many numbers from 1 to 1000 contain exactly two 5's among its digits?

40. How many positive integers less than 20,000 are both perfect squares and perfect cubes?