MATH FIELD DAY 2018

Contestants' Instructions
Team Essay

1. Your team has *forty* minutes to answer this set of questions.

2. All answers must be justified with complete explanations. Your answers should be clear, grammatically correct, and mathematically precise.

3. Your team may turn in at most one answer to each question. Number the answers and submit them in order, starting each problem on a separate page. You are allowed to skip any question and proceed on to later questions. Write on one side of the paper only, and number the pages you turn in consecutively. Do not turn in the problem packets. Write your school name on each page.

4. Your team may be organized in any way you choose. For example, you may designate one team member to write every answer, or you could designate different team members to write different answers. You may also use the chalk board.

5. Only official team members may be in the room during the essay period. Team members may <u>not</u> bring books, calculators, or any other materials into the room.

6. A proctor outside the room will warn you ten minutes before your time is up.

7. Give your answers to the proctor at the end of the period to place in the inner envelope. Please erase all chalk boards in the room and put all scratch paper in the trash can, leaving the room neat.

Counting Binary Trees and Triangulations

A graph consists of finitely many designated points (called vertices) and line segments (called edges) whose endpoints are vertices. A tree is a graph drawn in the plane that has a vertex P such that there is exactly one way we can reach every other vertex by following a sequence of edges downwards from P. We call P the root of the tree. A tree is binary if every vertex has 0 or 2 edges leading downwards from it. For example, Figure 1 shows a binary tree with 9 vertices.

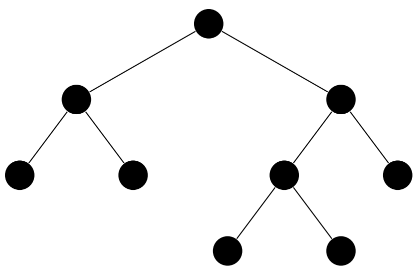

Figure 1

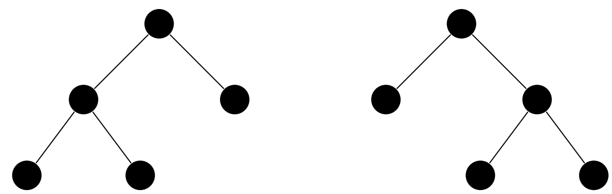

Figure 2

In distinguishing between binary trees, we care whether we travel downwards from a vertex to the left or to the right, but we don't care about the spacing of vertices. Thus, Figure 2 shows two different binary trees with five vertices, and they are the only binary trees with five vertices.

Problem 1. There are five binary trees with seven vertices. Draw them. No explanation is needed.

We will determine the number r_v of binary trees with v vertices for every positive integer v. For example, Figure 2 shows that $r_5 = 2$, and Problem 1 shows that $r_7 = 5$. We also have $r_1 = 1$ and $r_3 = 1$, since Figure 3 shows the only binary tree with one vertex, and Figure 4 shows the only binary tree with three vertices.

Figure 3

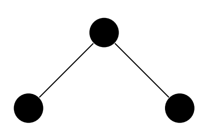

Figure 4

A binary tree T with more than one vertex consists of a root P and two edges leading downwards from P to the roots of binary trees T_1 and T_2 with fewer vertices than T (Figure 5). For example, the root in Figure 1 lies on edges leading downwards to the roots of the trees T_1 and T_2 in Figure 6.

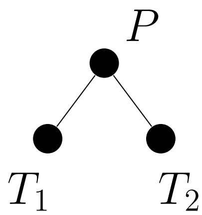

Figure 5

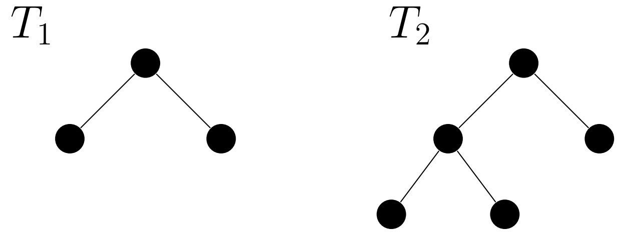

Figure 6

> **Problem 2.** Use the discussion of Figure 5 to show that every binary tree has an odd number of vertices.

Problem 2 shows that $r_v = 0$ for v even. We must still determine r_v for v odd, and so $v = 2k + 1$ for an integer $k \geq 0$. We can assume that $k \geq 1$ since we already know that $r_1 = 1$.

Consider the following assertion.

<u>Claim 1</u>: For any odd integer $v \geq 3$, r_v is the sum of the products $r_g r_h$ as (g, h) varies over all ordered pairs of odd positive integers g and h that sum to $v - 1$.

For example, since $r_1 = 1$, $r_3 = 1$, and $r_5 = 2$ (by the discussions of Figures 3, 4, and 2), Claim 1 states that

$$r_7 = r_1 r_5 + r_3 r_3 + r_5 r_1 = 1 \cdot 2 + 1 \cdot 1 + 2 \cdot 1 = 5,$$

in agreement with Problem 1.

> **Problem 3.** Use Claim 1 and the known values of r_1, r_3, r_5, and r_7 to find the value of r_9, the number of binary trees with nine vertices.

> **Problem 4.** Explain why Claim 1 follows from Problem 2 and the discussion of Figure 5.

Let $f(i)$ be a numerical expression in terms of an integer i. For any positive integer n, let

$$\sum_{i=1}^{n} f(i) = f(1) + f(2) + \cdots + f(n)$$

be the sum of the numbers $f(i)$ as i varies over the integers from 1 through n. For example, we have

$$\sum_{i=1}^{4} \frac{60}{i+1} = \frac{60}{2} + \frac{60}{3} + \frac{60}{4} + \frac{60}{5} = 30 + 20 + 15 + 12 = 77.$$

Problem 5. Explain why Claim 1 shows that

$$r_{2k+1} = \sum_{i=1}^{k} r_{2i-1} r_{2k-2i+1} \tag{1}$$

for every positive integer k.

A lattice point is a point of the (x, y) plane with integer coordinates. A path is a sequence of moves between lattice points such that each move increases one coordinate by 1 and leaves the other coordinate unchanged.

For any integer $k \geq 1$, let t_k be the number of paths from $(0, 0)$ to (k, k) such that the lattice points on each path lie on or below the line $y = x$. For example, we have $t_3 = 5$, since Figure 7 shows that there are five paths from $(0, 0)$ to $(3, 3)$ such that the lattice points on each path lie on or below the line $y = x$.

We set $t_0 = 1$ since only one path starts and remains at $(0, 0)$, namely, the path with no moves.

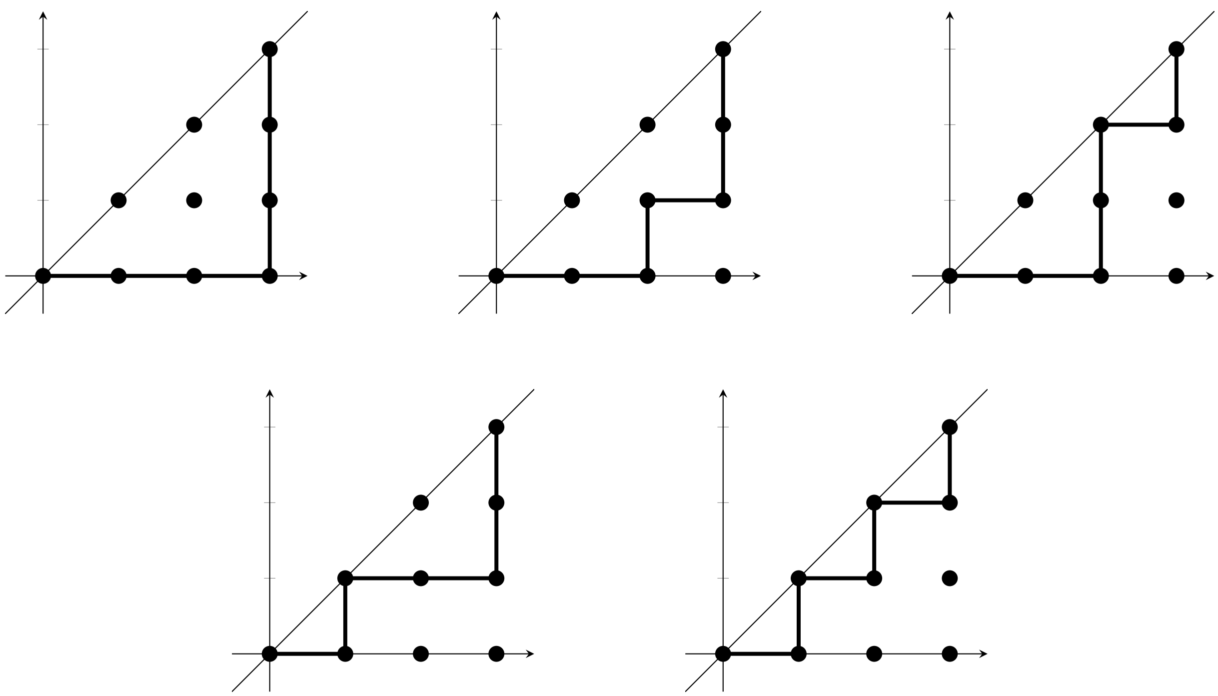

Figure 7

Figure 8 shows that there are five paths from $(0, 0)$ to $(4, 4)$ that lie below the line $y = x$ for $0 < x < 4$. That fact that Figure 7 and 8 show the same number of paths illustrates the next problem for $i = 4$.

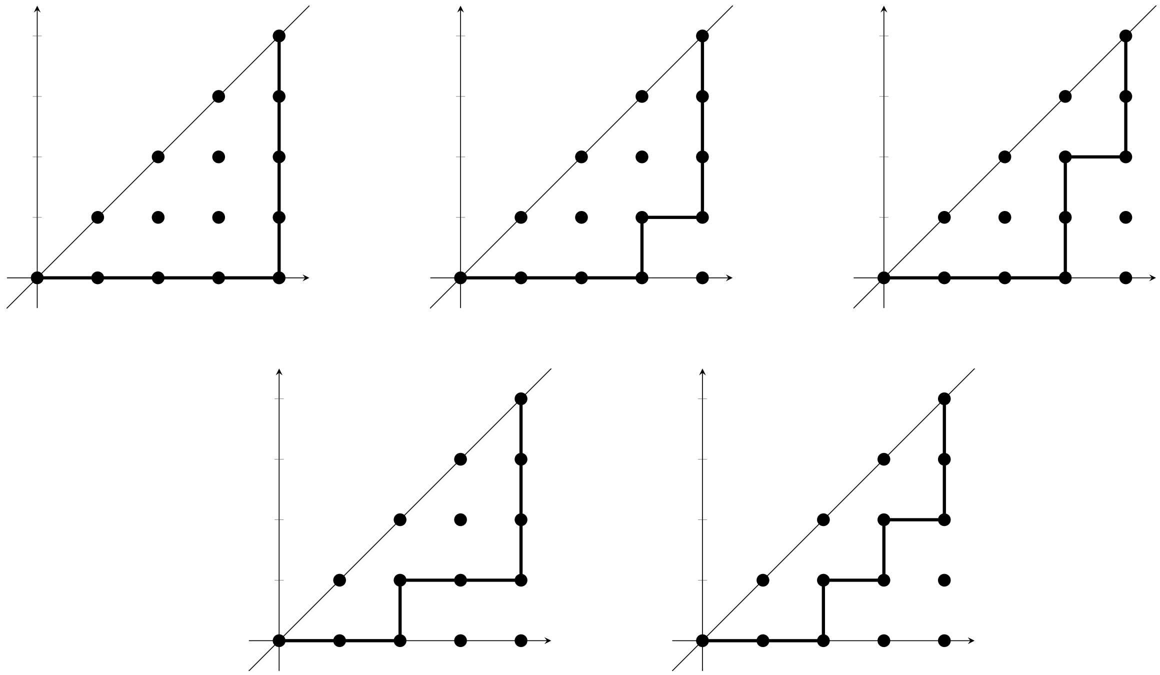

Figure 8

Problem 6. Let i be a positive integer. Explain why t_{i-1} is the number of paths from $(0,0)$ to (i,i) that lie below the line $y = x$ for $0 < x < i$.

Problem 7. Use Problem 6 to show that

$$t_k = \sum_{i=1}^{k} t_{i-1} t_{k-i} \tag{2}$$

for every positive integer k.

Problem 1 and Figure 7 show that r_7 and t_3 both equal 5. This illustrates Equation 3 below for $k = 3$.

Problem 8. Prove that

$$r_{2k+1} = t_k \tag{3}$$

for every integer $k \geq 0$. Use Problems 5 and 7 to argue recursively.

For any positive integer n and any integer r with $1 \leq r \leq n$, set

$$\binom{n}{r} = \frac{n(n-1)(n-2)\cdots(n-(r-1))}{r(r-1)\,(r-2)\,\cdots\,1}$$

where there are r factors in both the numerator and denominator on the right-hand side. Also set $\binom{n}{0} = 1$.

For every integer $k \geq 1$, applying the 2017 Team Essay to the line $y = x$ of slope $m = 1$ gives the formula

$$t_k = \frac{1}{k}\binom{2k}{k-1} \tag{4}$$

for the number of paths from $(0, 0)$ to (k, k) whose lattice points lie on or below the line $y = x$. Equations 3 and 4 determine the value of r_v for every odd integer $v \geq 3$.

Problem 9. Use Equations 3 and 4 to find the number of binary trees with 15 vertices. Simplify the answer.

The analogous equations 1 and 2 recursively count binary trees and paths along lattice points. A third analogous equation 5 in Problem 11 recursively counts triangulations.

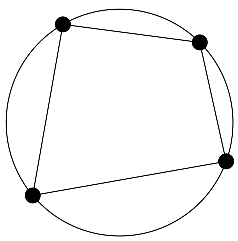

Figure 9

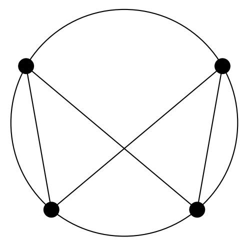

Figure 10

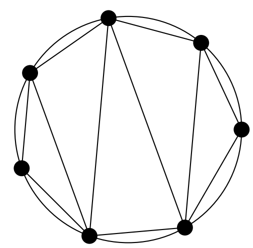

Figure 11

A polygon P is cyclic if it is inscribed in a circle S and its vertices occur in the same order on both P and S. For example, Figure 9 shows a cyclic quadrilateral, but Figure 10 does not. A triangulation of P is a choice of chords of S such that each chord joins two vertices of P, no two chords intersect inside S, and the chords divide up the interior of P into triangles. For example, Figure 11 shows a triangulation of a cyclic polygon with seven vertices.

All cyclic polygons with the same number of vertices have the same number of triangulations. For any integer $k \geq 1$, let w_k be the number of triangulations of a cyclic polygon with $k + 2$ vertices. For example, we have $w_3 = 5$, since a cyclic polygon with $3 + 2$ vertices – namely, a cyclic pentagon – has five triangulations (Figure 12). We have $w_1 = 1$ because the only triangulation of a cyclic polygon with 1+2 vertices – namely, a triangle – consists of the triangle itself. Set $w_0 = 1$ by definition.

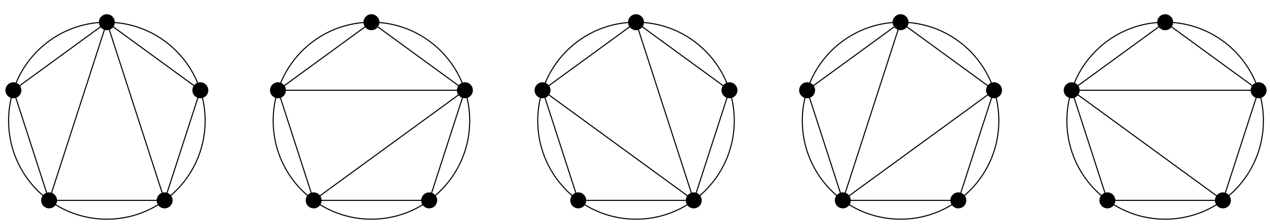

Figure 12

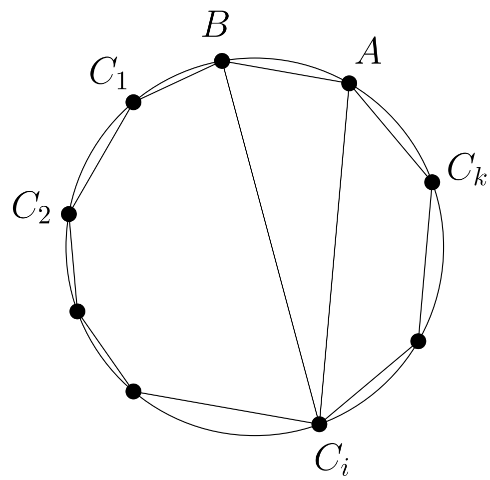

Figure 13

Problem 10. Let P be a cyclic polygon with $k+2$ vertices for a positive integer k. Label the vertices consecutively around P as

$$A, B, C_1, C_2, \ldots, C_k$$

(Figure 13). Let i be an integer from 1 through k. Show that

$$w_{i-1}w_{k-i}$$

is the number of triangulations of P that use ABC_i as one of the triangles dividing up the interior of P. Cases where $i = 1$ or $i = k$ may need separate consideration.

Problem 11. Deduce from Problem 10 that

$$w_k = \sum_{i=1}^{k} w_{i-1}w_{k-i} \tag{5}$$

for every positive integer k. Then use Equations 2, 4, and 5 to prove for every positive integer k that the number of triangulations of a cyclic polygon with $k+2$ vertices is

$$\frac{1}{k}\binom{2k}{k-1}.$$

A polygon P is <u>convex</u> if, for each edge e of P, all points of P not on e lie on one side of the line through e. The definition of triangulations and the results of Problems 10 and 11 generalize from cyclic to convex polygons.