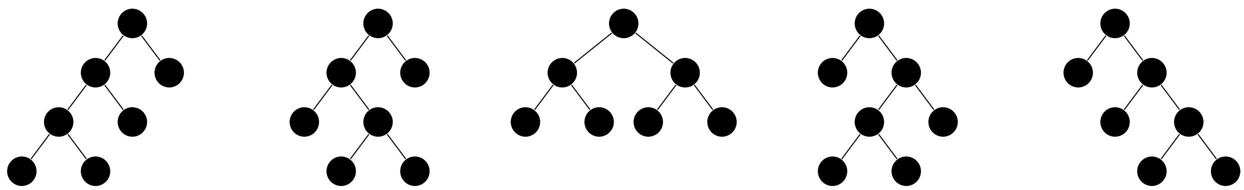
MATH FIELD DAY 2018
Team Essay Solutions

Problem 1. There are five binary trees with seven vertices. Draw them. No explanation is needed.

Solution.



□

Problem 2. Use the discussion of Figure 5 to show that every binary tree has an odd number of vertices.

Solution. Let $|T|$ denote the number of vertices in a binary tree T. If T has only one vertex, then $|T| = 1$, so $|T|$ is odd. If T has more than one vertex, then T consists of a root P and any additional vertices come in pairs, and so $|T|$ remains odd.

Alternatively, we can argue via recursion: P is connected to the roots of trees T_1 and T_2, each with fewer vertices than T, as in Figure 5. Then we have

$$|T| = |T_1| + |T_2| + 1. \tag{6}$$

Because T_1 and T_2 have fewer vertices than T, we know recursively that $|T_1|$ and $|T_2|$ are odd. Then $|T_1| + |T_2|$ is even, and so (6) shows that $|T|$ is odd. □

Problem 3. Use Claim 1 and the known values of r_1, r_3, r_5, and r_7 to find the value of r_9, the number of binary trees with nine vertices.

Solution. $r_9 = r_1r_7 + r_3r_5 + r_5r_3 + r_7r_1 = 1 \cdot 5 + 1 \cdot 2 + 2 \cdot 1 + 5 \cdot 1 = 14.$ □

Problem 4. Explain why Claim 1 follows from Problem 2 and the discussion of Figure 5.

Solution. A binary tree T with at least three vertices is determined by the left and right trees T_1 and T_2 whose roots are attached to the root P of T.

Let g and h be positive integers. There are $r_g r_h$ trees binary trees T where T_1 has g vertices and T_2 has h vertices (since there are r_g choices for T_1 and r_h choices for T_2, and each choice of T_1 can be paired with each choice of T_2). Because the vertices of T consist of the g vertices of T_1, the h vertices of T_2, and the root P of T, T has v vertices exactly when $g + h + 1 = v$, that is, when g and h sum to $v - 1$. By Problem 2, the positive integers g and h must be odd.

We get all binary trees T with v vertices as g and h vary over all pairs of odd positive integers that sum to $v - 1$. Adding the numbers $r_g r_h$ of possibilities in each case gives the total number r_v of possibilities for T. □

Problem 5. Explain why Claim 1 shows that

$$r_{2k+1} = \sum_{i=1}^{k} r_{2i-1} r_{2k-2i+1} \tag{1}$$

for every positive integer k.

Solution. Let k be a positive integer. Setting $v = 2k + 1$ makes v an odd integer and $v \geq 3$. By Claim 1, r_{2k+1} is the sum of the products $r_g r_h$ as (g, h) varies over all ordered pairs of odd positive integers g and h that sum to $v - 1 = 2k$. Each such g is less than $2k$ (since $g = 2k - h$ and $h > 0$). As i runs over the integers from 1 through k, $g = 2i - 1$ runs over the odd positive integers less than $2k$. For each value of i, $g + h = 2k$ holds exactly when

$$h = 2k - g = 2k - (2i - 1) = 2k - 2i + 1.$$

This value of h is an odd integer, and it is positive (since $i \leq k$). In short, we have

$$r_{2k+1} = \sum_{i=1}^{k} r_{2i-1} r_{2k-2i+1}.$$

□

Problem 6. Let i be a positive integer. Explain why t_{i-1} is the number of paths from $(0, 0)$ to (i, i) that lie below the line $y = x$ for $0 < x < i$.

Solution. A path from $(0,0)$ to (i,i) lies below the line $y=x$ for $0<x<i$ if and only if the path starts by moving horizontally from $(0,0)$ to $(1,0)$, ends by moving vertically from $(i,i-1)$ to (i,i), and in between follows a path from $(1,0)$ to $(i,i-1)$ whose lattice points lie below the line $y=x$. Because lattice points have integer coordinates and the line $y=x-1$ lies 1 unit below $y=x$, lattice points lie below $y=x$ if and only if they lie on or below $y=x-1$. Thus, we want to count the number of paths from $(1,0)$ to $(i,i-1)$ whose lattice points lie on or below the line $y=x-1$.

A path P runs from $(1,0)$ to $(i,i-1)$ if and only if its translation Q 1 unit to the left runs from $(0,0)$ to $(i-1,i-1)$. The lattice points of P lie on or below $y=x-1$ if and only if the lattice points of Q lie on or below $y=x$, since $y=x$ lies one unit to the left of $y=x-1$. Thus, we want to count the number of paths from $(0,0)$ to $(i-1,i-1)$ whose lattice points lie on or below $y=x$. This number is t_{i-1}. □

Problem 7. Use Problem 6 to show that

$$t_k=\sum_{i=1}^{k} t_{i-1}t_{k-i} \tag{2}$$

for every positive integer k.

Solution. Fix a positive integer k. Let S be the set of paths from $(0,0)$ to (k,k) such that the lattice points on each path lie on or below the line $y=x$. For each integer i from 1 through k, let S_i be the set of paths in S such that (i,i) is the first lattice point on the path after $(0,0)$ to lie on the line $y=x$. Because each path in S contains (k,k), it belongs to exactly one of the sets S_i. Thus we have

$$t_k=|S|=\sum_{i=1}^{k}|S_i|, \tag{7}$$

where we write $|X|$ for the number of elements in a set X.

Fix an integer i from 1 through k. To get all the paths in S_i, take any path Q from $(0,0)$ to (i,i) that lies below $y=x$ for $0<x<i$ and follow it with any path R from (i,i) to (k,k) whose lattice points lie on or below $y=x$. There are t_{i-1} choices for the path Q, by Problem 6. We get the possible paths R by taking the t_{k-i} paths from $(0,0)$ to $(k-i,k-i)$ whose vertices lie on or below $y=x$ and translating them i units up and i units to the right (since that translation maps the line $y=x$ to itself). Pairing each choice for Q with each choice of R shows that

$$|S_i|=t_{i-1}t_{k-i}.$$

Together with (7), this gives the desired equation

$$t_k=\sum_{i=1}^{k} t_{i-1}t_{k-i}.$$

□

Problem 8. Prove that

$$r_{2k+1} = t_k \tag{3}$$

for every integer $k \geq 0$. Use Problems 5 and 7 to argue recursively.

Solution. The equation

$$r_{2k+1} = t_k \tag{3}$$

holds for $k = 0$ because r_1 and t_0 both equal 1. To prove that (3) holds for a positive integer k, we can assume that we already know that

$$r_{2j+1} = t_j \tag{8}$$

for all integers j from 0 through $k - 1$. Problem 7 shows that

$$t_k = \sum_{i=1}^{k} t_{i-1} t_{k-i}.$$

As i runs over the integers from 1 through k, both $i - 1$ and $k - i$ run over the integers from 0 through $k - 1$, and so (8) shows that

$$t_{i-1} = r_{2(i-1)+1} = r_{2i-1}$$

and

$$t_{k-i} = r_{2(k-i)+1} = r_{2k-2i+1}.$$

Then substitution in (2) gives

$$t_k = \sum_{i=1}^{k} r_{2i-1} r_{2k-2i+1}.$$

This sum equals r_{2k+1} by Problem 5. □

Problem 9. Use Equations 3 and 4 to find the number of binary trees with 15 vertices. Simplify the answer.

Solution. Because $15 = 2k + 1$ for $k = 7$, Problem 8 shows that the number r_{15} of binary trees with 15 vertices is t_7. By equation 4, this number is

$$\frac{1}{7}\binom{14}{6} = \frac{1}{7} \cdot \frac{14 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 13 \cdot 11 \cdot 3 = 429.$$

□

Problem 10. Let P be a cyclic polygon with $k+2$ vertices for a positive integer k. Label the vertices consecutively around P as

$$A, B, C_1, C_2, \ldots, C_k$$

(Figure 13). Let i be an integer from 1 through k. Show that

$$w_{i-1}w_{k-i}$$

is the number of triangulations of P that use ABC_i as one of the triangles dividing up the interior of P. Cases where $i = 1$ or $i = k$ may need separate consideration.

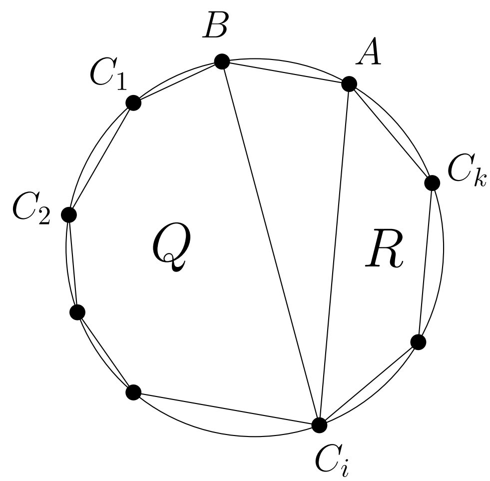


Figure 14

Solution. Suppose first that $2 \leq i \leq k-1$ (Figure 14). Because the list

$$B, C_1, C_2, \ldots, C_i$$

contains $i+1$ consecutive vertices of P, they are the vertices of a cyclic polygon Q (since $i+1 \geq 3$ for $i \geq 2$). Q has w_{i-1} triangulations (since $i+1$ is 2 more than $i-1$). Because the list

$$C_i, C_{i+1}, \ldots, C_k, A$$

contains $k-i+2$ consecutive vertices of P, they are the vertices of a cyclic polygon R (since $k-i+2 \geq 3$ for $i \leq k-1$). R has w_{k-i} triangulations (since $k-i+2$ is 2 more than $k-i$). Combining each triangulation of Q with each triangulation of R and attaching triangle ABC_i gives $w_{i-1}w_{k-i}$ triangulations of P. These are all the triangulations of P that use triangle ABC_i because no such triangulation has a chord intersecting the interior of triangle ABC_i.

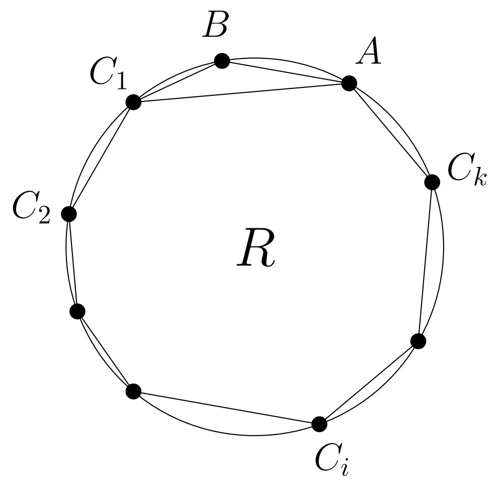


Figure 15

Secondly, suppose that $i = 1$ and $k \geq 2$. (Figure 15). Because the list

$$C_1, C_2, \ldots, C_k, A$$

contains $k + 1$ consecutive vertices of P, they are the vertices of a cyclic polygon R (since $k + 1 \geq 3$ for $k \geq 2$). R has w_{k-1} triangulations (since $k + 1$ is 2 more than $k - 1$). Attaching triangle ABC_1 to each one gives the triangulations of P that use triangle ABC_1 (since no such triangulation has a chord intersecting the interior of triangle ABC_1). We get the number w_{k-1} of these triangulations by setting $i = 1$ in $w_{i-1}w_{k-i}$ (since $w_0 = 1$).

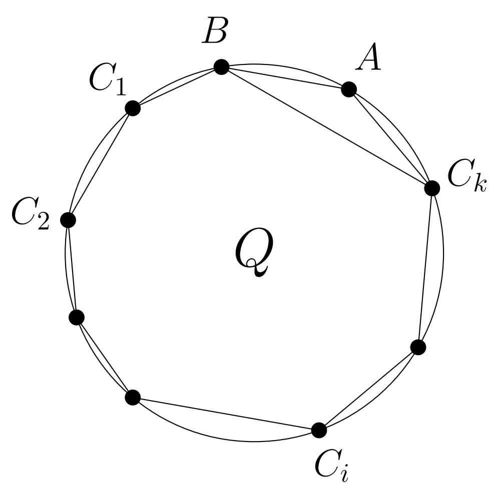


Figure 16

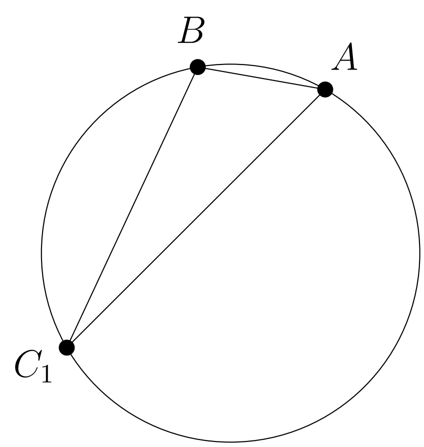


Figure 17

Suppose next that $i = k$ and $k \geq 2$. (Figure 16). Because the list

$$B, C_1, C_2, \ldots, C_k$$

contains $k + 1$ consecutive vertices of P, they are the vertices of a cyclic polygon Q (since $k + 1 \geq 3$ for $k \geq 2$). Q has w_{k-1} triangulations (since $k + 1$ is 2 more than $k - 1$). Attaching triangle ABC_k to each one gives the triangulations of P that use triangle ABC_k (since no such triangulation has a chord intersecting the interior of triangle ABC_k). We get the number w_{k-1} of these triangulations by setting $i = k$ in $w_{i-1}w_{k-i}$ (since $w_0 = 1$).

Finally, suppose that $k = 1$, and so $i = 1$. (Figure 17). P is triangle ABC_1, whose only triangulation is itself. We get the number 1 of triangulations by setting $k = 1$ and $i = 1$ in $w_{i-1}w_{k-i}$ (since $w_0 = 1$). □

Problem 11. Deduce from Problem 10 that

$$w_k = \sum_{i=1}^{k} w_{i-1}w_{k-i} \tag{5}$$

for every positive integer k. Then use Equations 2, 4, and 5 to prove for every positive integer k that the number of triangulations of a cyclic polygon with $k + 2$ vertices is

$$\frac{1}{k}\binom{2k}{k-1}.$$

Solution. Let k be a positive integer, and let P be a cyclic polygon with $k + 2$ vertices. Label consecutive vertices $A, B, C_1, \ldots, C_k$ as in Problem 10. In order to divide the interior of P into nonoverlapping triangles, every triangulation of P uses exactly one triangle with edge AB, and so it uses triangle ABC_i for exactly one integer i from 1 through k. Thus, the total number w_k of triangulations of P is the sum of the numbers that use triangle ABC_i for $1 \le i \le k$, and those numbers are $w_{i-1}w_{k-i}$ (by Problem 10). This gives the equation

$$w_k = \sum_{i=1}^{k} w_{i-1}w_{k-i}. \tag{5}$$

We prove recursively that

$$w_k = t_k \tag{9}$$

for each integer $k \ge 0$. Equation 9 holds for $k = 0$ because w_0 and t_0 both equal 1. To prove (9) for a positive integer k, we can assume we already know that

$$w_j = t_j \tag{10}$$

for all integers j from 0 through $k - 1$. As i runs over the integers from 1 through k, both $i - 1$ and $k - i$ run over the integers from 0 through $k - 1$, and so (10) holds when j is $i - 1$ or $k - i$. Thus, we have

$$\begin{aligned} w_k &= \sum_{i=1}^{k} w_{i-1}w_{k-i} && \text{(by (5))} \\ &= \sum_{i=1}^{k} t_{i-1}t_{k-i} && \text{(by (10))} \\ &= t_k, && \text{(by (2))} \end{aligned}$$

completing the proof of (9).

For every positive integer k, Equation 4 says that t_k equals

$$\frac{1}{k}\binom{2k}{k-1}.$$

This also equals the number w_k of triangulations of a cyclic polygon with $k+2$ vertices, by (9). □