Solutions to 2001 Team Essay

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                1               1               0
   1.  (a)     0 0        (b)  1 1       (c)   0 0
                1               1               0
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2. Transforming Figure 3 repeatedly gives the following arrangements.

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(a)   1             (b)   1          (c)   0          (d)   1
     0 1                 1 0              1 1              0 1
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Because (d) is the same as (a), the arrangements (a)-(c) repeat in order forever. For example, the next arrangement after (d) is the same as (b) (since (d) is the same as (a)), the arrangement after that is the same as (c), the one after that is the same as (a), and so on. Since none of these arrangements has all zeroes, we never get all zeroes.

3. If we start with two zeroes, we are done. If we start with two ones, the next arrangement is two zeroes. If we start with a one and a zero, the next arrangement is two ones, and the arrangement after that is two zeroes.

4. First suppose that we start with the same bit at A and C. If the same bit also appears at B, the next arrangement has zeroes at A and B, and the arrangement after that has a zero at A. If the starting bit at B is different from the one at A and C, then the next arrangement has ones at A and B, and the arrangement after that has a zero at A. In either case, we end up with a zero at A after two transformations if we start with the same bit at A and C.

Next suppose that we start with different bits at A and C. Exactly one of these bits agrees with the starting bit at B, and so the next arrangement has a zero and a one in some order at A and B. Then the next arrangement has a one at A, and so we end up with a one at A after two transformations when we start with different bits at A and C.

5. By Problem 4, applying two transformations to the given arrangement has the same effect as applying one transformation to each of the two arrangements formed by taking every other bit. Doing this twice shows that applying four transformations has the same effect as applying one transformation to each of the four arrangements formed by taking every fourth bit. Continuing in this way shows that applying 2^(k-1) transformations has the same effect as applying one transformation to each of the 2^(k-1) arrangements of two bits formed by taking every 2^(k-1) th bit. Each arrangement of two bits becomes all zeroes in at most two transformations, by Problem 3. Thus the original arrangement becomes all zeroes after at most 2x2^(k-1) = 2^k transformations.

6. As we move counterclockwise around the given

arrangement, whenever the bits change parity between zero and one we get a one in the next arrangement. When we've traveled once around the given arrangement, we've changed parity an even number of times, since we end with the same bit as we started. Thus the transformed arrangement has an even number of ones.

7. If an arrangement is not all zeroes ok all ones, the bits change parity between zero and one as we move counterclockwise around the arrangement. This gives a pair of consecutive bits consisting of a zero and a one, which leads to a one in the next arrangement. Thus the only arrangements that can transform into all zeroes are those that are all zeroes or all ones.

8. Suppose that we start with an arragement of an odd number of bits that are not all zeroes or all ones. Applying a transformation gives a second arrangement, which is not all zeroes, by Problem 7. It is not all ones, since it has an even-number of ones (by Problem 6) and an odd number of bits.

We now know that the second arrangement is not all zeroes or all ones. Applying another transformation and repeating the argument in the last paragraph shows that the third arrangement is not all zeroes or all ones. Continuing in this way shows that we never get all zeroes (or all ones) unless we start with all zeroes or all ones, provided that there are an odd number of bits.

9. When the total number of bits in an arrangement is not a power of 2, it has the form $(2^r)s$ for an odd integer $s>1$ and an integer $r>=0$. As in the solution to Problem 5, applying 2^r transformations to the given arrangement is the same as applying one transformation to each of the 2^r arrangements of s bits formed by taking every 2^r th bit. One of these arrangements has exactly one 1. Because this arrangement has an odd number $s>l$ of bits and is not all zeroes or all ones, it does not become all zeroes after any number of transformations (by Problem 8). Thus, the given arrangement of $(2^r)s$ bits does not become all zeroes after $(2^r)t$ transformations for any positive integer t. It follows that the given arrangement never becomes all zeroes because, if it did, it would stay all zeroes under additonal transformations until we've applied a multiple of 2^r transformations in all.