Reflex Vertices

For any integer $n \geq 3$, the polygon determined by the n points

$$A_1, \ldots, A_n \quad (1)$$

in the plane consists of the n segments

$$A_1A_2, A_2A_3, \ldots, A_{n-1}A_n, A_nA_1, \quad (2)$$

provided that we never pass through a point more than once in following the segments in (2) in order from A_1 to A_n and back to A_1. We refer to the points in (1) as vertices, the segments in (2) as sides. Note that we require polygons not to pass through points more than once.

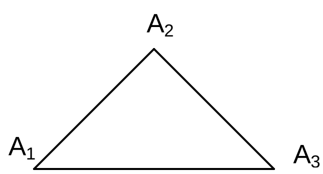

Figure 1

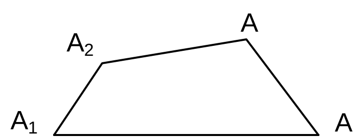

Figure 2a

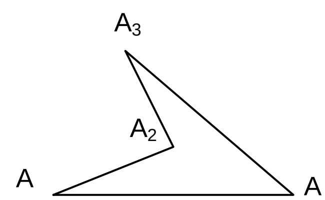

Figure 2b

A polygon with 3 sides is a triangle. (Figure 1) Figures 2a and 2b are polygons with 4 sides, or quadrilaterals. Figures 3a and 3b are not polygons because they pass through points twice.

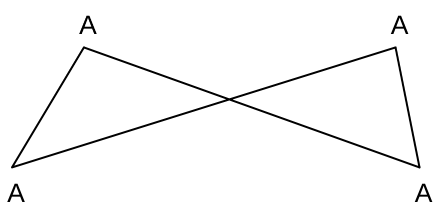

Figure 3a

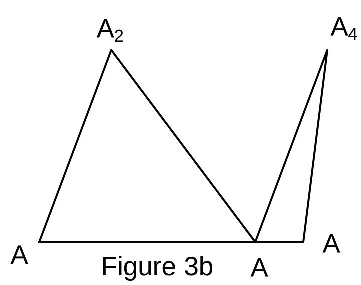

Figure 3b

A reflex vertex of a polygon is a vertex where the polygon has an internal angle greater than 180^0. For example, the reflex vertices of the polygon in Figure 4 are labeled T, U, and V. The internal angles at these vertices are marked by circular arcs greater than semicircles in Figure 5.

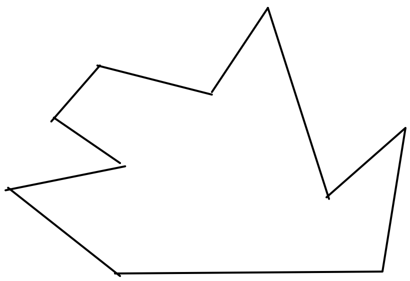

Figure 4

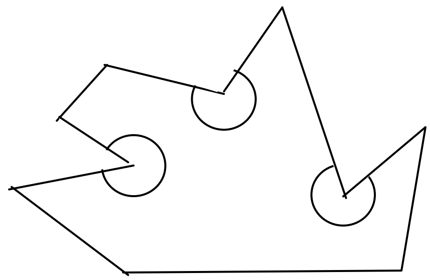

Figure 5

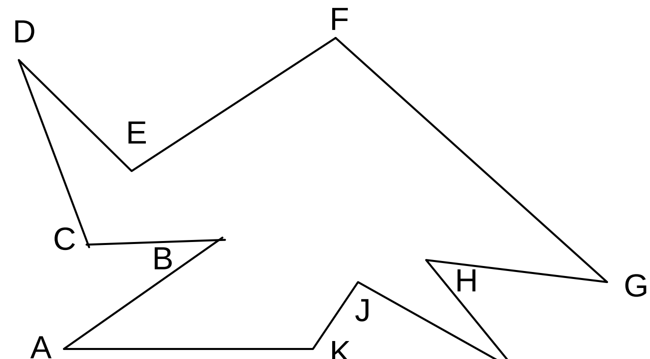

Figure 6

Problem 1. What are the letters of the reflex vertices in Figure 6?

Let T be a reflex vertex of a polygon P.

Problem 2. Why is there a ray s that originates at T, divides the internal angle of P at T into two angles less than 180^o, and does not contain any vertex of P other than T? (Figure 7)

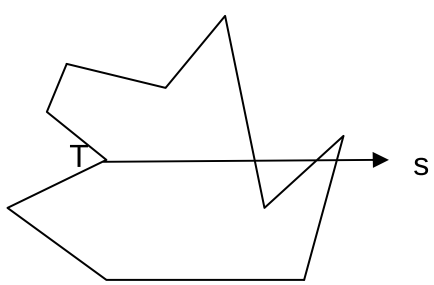

Figure 7

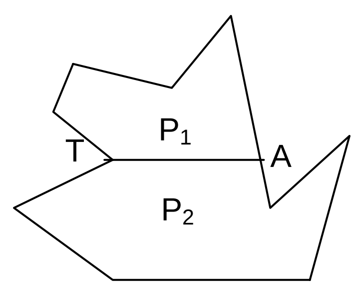

Figure 8

Because the ray s proceeds from T through the interior of P, it intersects P in at least one point other than T, and one such point A lies closest to T. (Figure 8) Segment TA divides the polygon P into two polygons P_1 and P_2 having only one side TA in common. In Figure 8, P_1 and P_2 are the two smaller polygons consisting of segment TA and the top and bottom portions of P.

Problem 3. Any reflex vertex of P other than T is a reflex vertex of P_1 or P_2 but not both. Why are these the only reflex vertices of P_1 and P_2 ? In other words, why can't P_1 or P_2 have a reflex vertex at T, A, or a vertex of P other than a reflex vertex?

We call the process of replacing P with P_1 and P_2 dividing P at T.

If P_1 or P_2 has a reflex vertex, we divide it there. For example, dividing the upper polygon in Figure 8 at the reflex vertex labeled U in Figure 9 gives Figure 10.

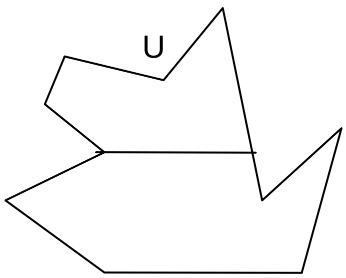

Figure 9

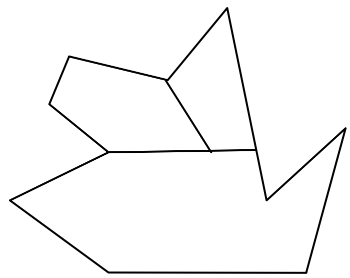
Figure 10

We continue in the same way until no reflex vertices are left. Dividing the lower polygon in Figure 10 at the reflex vertex labeled V in Figure 11 gives Figure 12.

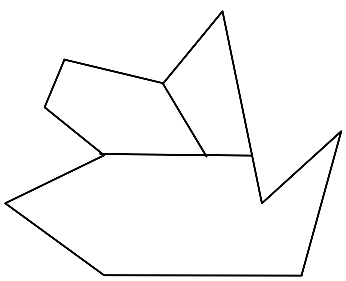
Figure 11

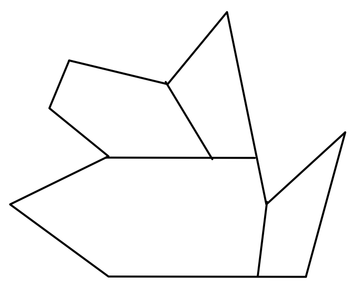
Figure 12

Because the four small polygons in Figure 12 have no reflex vertices, the process ends. In short, having started with the polygon in Figure 4 which has three reflex vertices, we divide polygons three times to get the four small polygons in Figure 12, which have no reflex vertices. The next problem generalizes this result.

Problem 4. If the original polygon P has $r \geq 1$ reflex vertices, conclude from Problem 3 that dividing polygons r times gives r + 1 polygons that have no reflex vertices.

Problem 5. Conclude from Problem 3 that each of the r + 1 polygons in Problem 4 contains at least one of the reflex vertices of the original polygon P. For example, the two upper polygons in Figure 12 contain the reflex vertex of P labeled U in Figure 4, and the two lower polygons in Figure 12 contain the reflex vertex of P labeled V in Figure 4.

Problem 6. Use the process above to divide the polygon in Figure 6 into polygons that have no reflex vertices. Show each division on a separate copy of the polygon, as in Figures 8, 10, and 12. Tear off the last page of this booklet, use the copies of Figure 6 there, and turn in that page to answer this problem.

Imagine that the floor plan of an art gallery is a polygon without interior walls. Guards are to be stationed in positions so that every point in the gallery is visible from the assigned position of at least one guard.

The following result was proved by B. Chazelle as part of a 1980 Yale Ph.D. thesis.

Theorem. An art gallery with $r \geq 1$ reflex vertices can be guarded by r people stationed at the reflex vertices.

For example, the art gallery in Figure 2b can be guarded by one person stationed near the only reflex vertex A_2. The gallery in Figure 4 can be guarded by three people stationed near the three reflex vertices T, U, and V.

Chazelle's Theorem concerns art galleries with at least one reflex vertex. Its proof depends on the following observation about art galleries with no reflex vertices: an art gallery with no reflex vertices can be guarded by one person stationed anywhere in the gallery or along its walls. For example, because the art gallery in Figure 2a has no reflex vertices, all of it is visible from any one position inside the polygon or along its sides.

Problem 7. Why does Chazelle's Theorem follow from Problem 5 and the observation underlined in the previous paragraph?

Although Chazelle's Theorem says that r guards are sufficient for any art gallery with $r \geq 1$ reflex vertices, some such galleries require fewer than r guards. For example, the art gallery in Figure 4 has three reflex vertices, but it can be guarded by one person stationed by vertex V. On the other hand, Figure 13 shows an art gallery that has three reflex vertices and that cannot be guarded by fewer than three people.

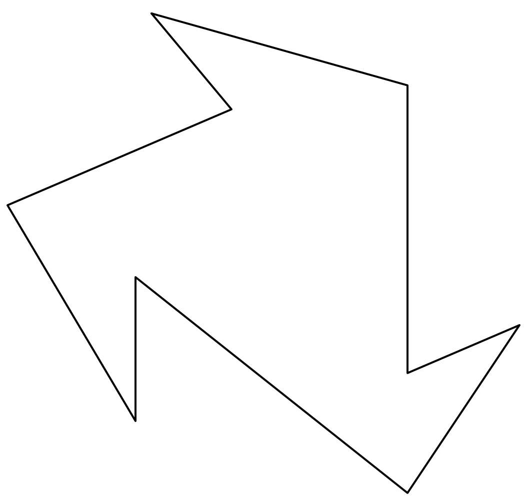

Figure 13

Problem 8. Show that, for any integer $r \geq 1$, there is an art gallery that has r reflex vertices and that cannot be guarded by fewer than r people.

Problem 8 shows that Chazelle's Theorem gives the best possible result for the number of guards that an art gallery requires in terms of the number of reflex vertices.

USE THE COPIES OF FIGURE 6 BELOW TO ANSWER PROBLEM 6.

TEAR OFF THIS PAGE AND TURN IT IN AS YOUR ANSWER TO PROBLEM 6.