<u>Bits in a Circle</u>

The digits 0 and 1 are called <u>bits</u>. Consider an arrangement of bits in a circle, where the number of bits is finite and at least two. We <u>transform</u> this arrangement into another as follows: we replace a bit with 0 if it is the same as the next bit counterclockwise, and we replace a bit with 1 if it is different from the next bit counterclockwise.

For example, suppose that we start with the arrangement in Figure 1. We have labeled the positions of the bits with the letters A-D for reference. This arrangement transforms as follows into the one in Figure 2. Because Figure 1 has different bits at A and B, Figure 2 has a 1 at A. Since Figure 1 has the same bits at B and C, Figure 2 has a 0 at B. Because Figure 1 has the same bits at C and D, Figure 2 has a 0 at C. Finally, since Figure 1 has different bits at D and A, Figure 2 has a 1 at D.

Problem 1. What arrangements result from transforming Figure 2

(a) once, (b) twice, and (c) three times?

Problem 2. Show that transforming Figure 3 any number of times never gives all zeroes.

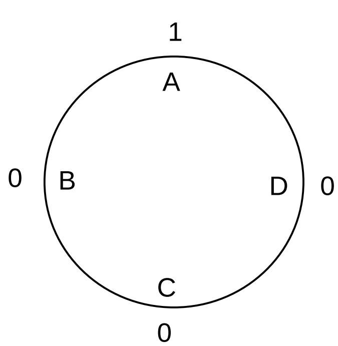

Figure 1

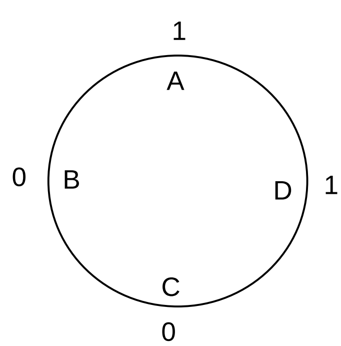

Figure 2

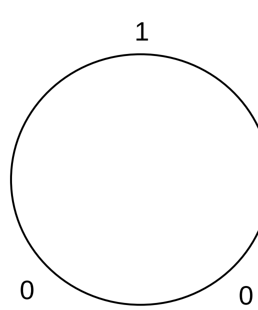

Figure 3

For what integers n do all arrangements of n bits eventually transform into all zeroes? This essay leads to the answer. Note that once an arrangement is all zeroes, it remains all zeroes through all further transformations.

Problem 3. Show that every arrangement of two bits becomes all zeroes after at most two transformations.

Problem 4. Let A, B, and C, labeled counterclockwise, be three consecutive positions of bits in an arrangement, which may have any number of other bits (Figure 4). Show that transforming the arrangement twice gives a 0 or 1 at A depending on whether or not the arrangement has the same bits at A and C.

Problem 4 shows that applying two transformations to an arrangement with an even number of bits has the same effect as applying one transformation to each of the two arrangements we get by taking every other vertex of the given arrangement. For example, suppose we start with an arrangement of six bits u – z (Figure 5). Applying two transformations to this arrangement has the same effect as applying one transformation to each of the two arrangements in Figure 6.

Problem 5. Use the preceding discussion and Problem 3 to show that 2^k transformations take every arrangement of 2^k bits to all zeroes for every positive integer k.

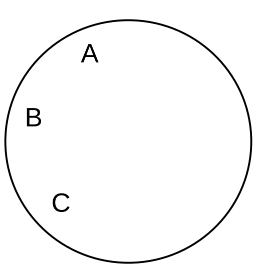

Figure 4

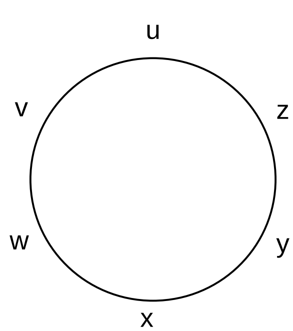

Figure 5

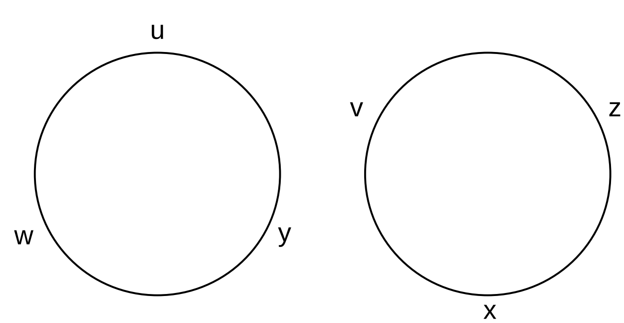

Figure 6

Problem 6. Prove that any arrangement transforms into one that has an even number of 1's. (Hint: Consider how the bits of the given arrangement alternate between 0's and 1's as one moves counterclockwise around the circle.)

Problem 7. Why are the arrangements that are all zeroes or all ones the only arrangements that give all zeroes after one transformation?

Problem 8. If an arrangement has an odd number of bits and gives all zeroes after any number of transformations, prove that the original arrangement has either all zeroes or all ones. Use Problems 6 and 7.

Problem 9. If an arrangement has exactly one 1 and a total number of bits that is not a power of two, prove that we never get all zeroes by applying any number of transformations. Use Problem 8 and the discussion after Problem 4. (Hint: If 2^r is the largest power of 2 dividing the number of bits, consider the effect of applying $2^r t$ transformations for any positive integer t.)

Problems 5 and 9 show that the powers of two are the only positive integers n such that every arrangement of n bits will eventually be transformed into all zeroes.