Solutions for the Team Essay on Reflex Vertices

1. B, E, H, J.

2. Because the internal angle at T is less than 360°, cutting it in half with a ray s gives two angles less than 180°. Rotating s slightly about T keeps both angles less than 180° while allowing infinitely many positions for s. Because only finitely many rays from T contain other vertices, there are infinitely many rays from T that do not contain any vertex other than T and that divide the internal angle at T into two angles less than 180°.

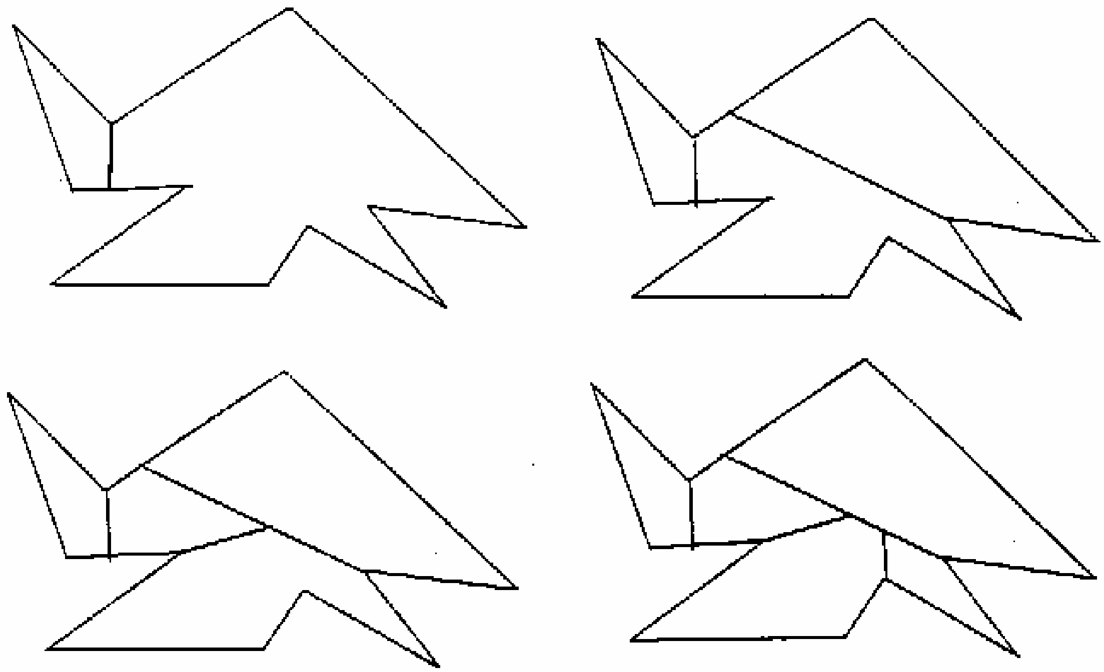
3. P_1 and P_2 do not have T as a reflex vertex because we divided the angle of P at T into two angles less than 180°. P_1 and P_2 have angles less than 180° at A because these angles sum to 180° (since A lies on a side of P and is not a vertex). Any angle of P that is less than 180° has the same measure as an angle of P_1 or P_2. Thus, the only reflex vertices of P_1 and P_2 are the reflex vertices of P other than T.

4. By Problem 3, P_1 and P_2 have reflex vertices at the reflex vertices of P other than T. P_1 and P_2 have no reflex vertices in common (since their only common vertices are T and A). Thus, P_1 and P_2 together have one less reflex vertex than P.

Each successive division likewise reduces the total number of reflex vertices by 1. Thus, having started with r reflex vertices, we reduce their number to 0 with r divisions. Each division increases the number of polygons by 1 (since it divides one polygon into two). Thus, since we started with 1 polygon, r divisions produce $r + 1$ polygons with no reflex vertices.

5. In the discussion accompanying Figures 7 and 8, we divided the polygon P at T into polygons P_1 and P_2, which both had T as a vertex. Likewise, each time we get two new polygons from an existing one by dividing it at a reflex vertex, both of the new polygons contain the reflex vertex. Thus, all the polygons produced by dividing at reflex vertices contain reflex vertices.

6.



7. Consider an art gallery P with $r \geq 1$ reflex vertices. Suppose that we station r guards at the reflex vertices. By Problem 4, the gallery P can be divided into r + 1 subpolygons $P_1, \ldots, P_{r+1}$ that have no reflex vertices.

Consider one of these subpolygons P_i. Because a vertex of P_i is a reflex vertex of P (by Problem 5), this vertex has a guard. The guard can see all of P_i (by the underlined observation before Problem 7, since P_i has no reflex vertices). Thus, the r guards cover all of the subpolygons $P_1, \ldots, P_{r+1}$ that comprise P, and so the guards cover all of P.

8. Suppose first that $r \geq 3$. Consider a regular polygon with r vertices. Move clockwise around the polygon and open up the last third of each side into an angle tipped so that it intercepts only the first third of the next side. (For example, Figure 13 and Figures A and B below show the cases where r is 3, 4, or 5.)

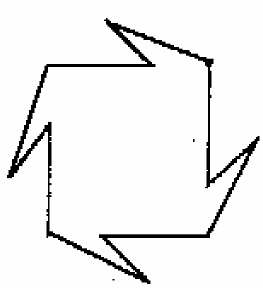


Figure A

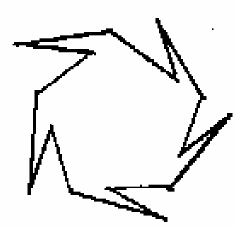


Figure B

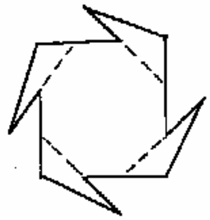


Figure C

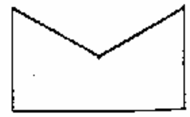


Figure D

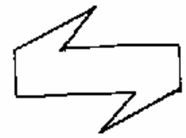


Figure E

The added angles have extensions into the polygon that do not intersect (as shown by the dotted lines in Figure C, which shows the same gallery as Figure A). Each added angle can be seen completely only by a guard inside the angle or its extension into the polygon. Because these extensions do not intersect, there must be a different guard inside each added angle or its extension. Thus, at least r guards are required. The gallery has r reflex vertices, one at the first edge of each added angle.

The case r = 1 is obvious because any gallery requires at least one guard (Figure D). For r = 2, Figure E shows a rectangle with two added angles tipped so that their extensions into the rectangle do not intersect. This gallery has 2 reflex vertices and requires at least 2 guards, one inside each added angle or its extension into the rectangle.