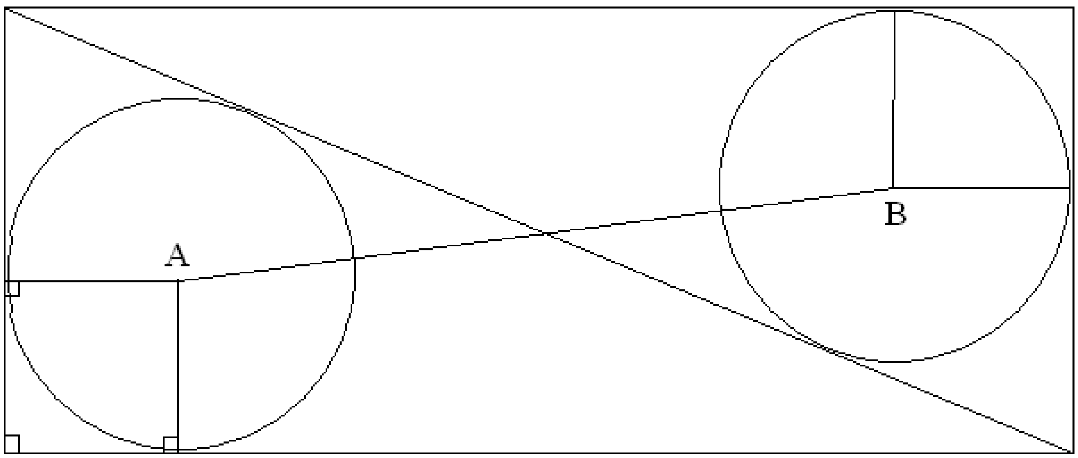
Huddle 2004 Solutions

1. The rectangle below is 12×5. The diagonal forms two triangles, and a circle is inscribed in each triangle. Find the distance between the centers of the two circles.



Solution: Each triangle has area $\frac{1}{2} \cdot 12 \cdot 5 = 30$, and semiperimeter $(5+12+13)/2 = 15$, so the circles each have radius $A/s = 2$. Set up a coordinate system with the origin at the lower lefthand vertex of the square. The center A will be the point $(2,2)$, and the center B will be $(12-2, 5-2) = (10,3)$, so the distance between the centers is $\sqrt{(2-10)^2 + (2-3)^2} = \sqrt{65}$.

2. The function D, whose domain is the set of positive integers, satisfies the conditions $D(p) = 1$ for all prime numbers p, and $D(mn) = mD(n) + nD(m)$ for all m, n. Evaluate $D(2004)$.

Solution: The prime factorization of 2004 is $2004 = 2^2 \cdot 3 \cdot 167$. Now $D(4) = D(2^2) = 2D(2) + 2D(2) = 4$, $D(12) = D(2^2 \cdot 3) = 4 \cdot 1 + 3 \cdot 4 = 16$, and $D(2004) = D(12 \cdot 167) = 12 \cdot 1 + 167 \cdot 16 = 2684$.

3. What is the remainder when the polynomial $p(x) = x^{1001}$ is divided by the polynomial $q(x) = x^2 - 1$?

Solution: Say $x^{1001} = q(x)(x^2-1) + r(x)$, where $r(x)$ has degree 1 or less. Then $r(1) = p(1) = 1$ and $r(-1) = p(-1) = -1$, so $r(x) = x$.

4. Suppose $F(x,y) = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2} \right)$. Evaluate and simplify $F(F(x,y))$.

Solution: $F(F(x,y)) = \left(\frac{\frac{x}{x^2+y^2}}{(\frac{x}{x^2+y^2})^2 + (\frac{y}{x^2+y^2})^2}, \frac{\frac{y}{x^2+y^2}}{(\frac{x}{x^2+y^2})^2 + (\frac{y}{x^2+y^2})^2} \right) = \left(\frac{x(x^2+y^2)}{x^2+y^2}, \frac{y(x^2+y^2)}{x^2+y^2} \right) = (x,y)$.