Huddle 2009

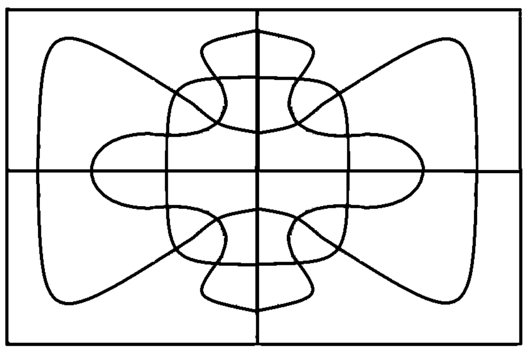
1. Find two integers greater than 1 whose product is 101010101.

Solution 1: $101010101 = 1 + 10^2 + 10^4 + 10^6 + 10^8 = \frac{10^{10}-1}{10^2-1} = \frac{10^5-1}{9} \cdot \frac{10^5+1}{11} = 11111 \times 9091$.

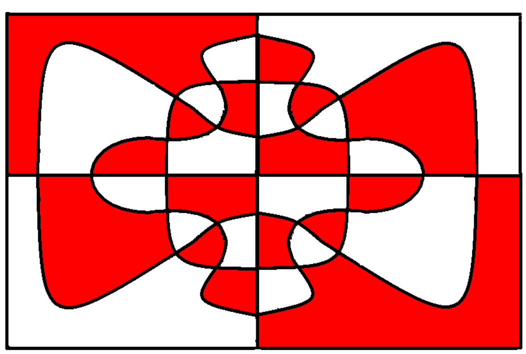
Solution 2: $11 \times 101010101 = 1111111111$. These 10 1's can be broken into two groups of 5, which yields the factors $1111111111 = 11111 \times 100001$. Since 11111 leaves a remainder of 1 when divided by 11, 100001 must be a multiple of 11, and indeed, long division shows that $100001 = 11 \times 9091$. Therefore $101010101 = 11111 \times 9091$.

(Other correct answers are $372,731 \times 271$ and $2,463,661 \times 41$.)

2. Suppose you are required to color all of the regions in the figure below in such a way that no two regions which meet at more than a single point get the same color. What is the least number of colors you could use?



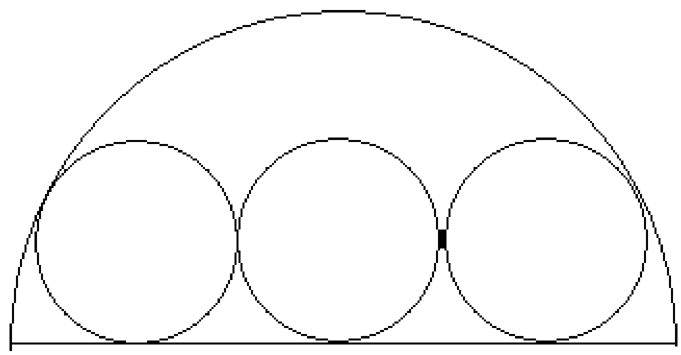
Solution: Two!



3. If $r > 0$ and $r^3 - 6r = 4$, evaluate $r^2 - 2r$.

Solution: $r^3 - 6r - 4 = (r^2 - 2r - 2)(r + 2) = 0$, so either $r = -2$ (impossible), or $r^2 - 2r = 2$.

4. Three small circles, each of radius 1, lie inside a larger semicircle. Each samll circle is tangent to the diameter of the large semicircle. The two outer small circles are each tangent to the semicircle, and the inner small circle is tangent to the other two. What is the radius of the large semicircle?



Solution: Let $A$ and $D$ be centers of circles, and $B$ and $C$ points of tangency, as shown below. Then the line perpendicular to the circles at common point of tangency $C$ passes throught $D$ and $A$. By the Pythagorean Theorem, $AD$ has length $\sqrt{5}$ and $DC$ has length 1, so radius $AC$ has length $1 + \sqrt{5}$.