2012 HUDDLE PROBLEMS
WITH SOLUTIONS

Problem 1. Given a positive integer n, which numbers in the set $S = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ could be the ones digit of n^2, given that the tens digit of n^2 is odd?

Solution. Write $n = 10a + b$. Then $n^2 = 100a^2 + 20ab + b^2$ has even tens digit unless the tens digit of b^2 is odd. By a quick enumeration, this leaves only the possibilities $b = 4$ and $b = 6$. In either case, b^2 has ones digit 6, and hence so does n^2. Thus the answer is only $\boxed{6.}$ □

Problem 2. Suppose that the numbers $1, 2, 3, \ldots, 2012$ are placed evenly around the outside of a circle in such a way that adjacent numbers differ by at most 2. What number is directly across from the number 361?

Solution. The only way to do this is to start with 1 some place, and list the even numbers consecutively in one direction, and the odd numbers in the other. Since 1 is across from 2012, opposite vertices add up to $2012 + 1$, so opposite from 361 is $2013 - 361 = \boxed{1652.}$ □

Problem 3. Let

$$S = \frac{1}{\log_{\frac{2}{3}}(2)} + \frac{1}{\log_{\frac{3}{4}}(2)} + \frac{1}{\log_{\frac{4}{5}}(2)} + \cdots + \frac{1}{\log_{\frac{2011}{2012}}(2)}$$

Find 2^S.

Solution 1. For any $b > 0$, we have $\frac{1}{\log_b 2} = \frac{1}{\frac{\log 2}{\log b}} = \frac{\log b}{\log 2} = \log_2 b$, and use this to re-write each summand:

$$S = \log_2\left(\frac{2}{3}\right) + \log_2\left(\frac{2}{3}\right) + \cdots + \log_2\left(\frac{2}{3}\right) = \log_2\left(\frac{2}{3} \times \frac{3}{4} \times \cdots \times \frac{2011}{2012}\right) = \log_2\left(\frac{2}{2012}\right) = \log_2\left(\frac{1}{1006}\right),$$

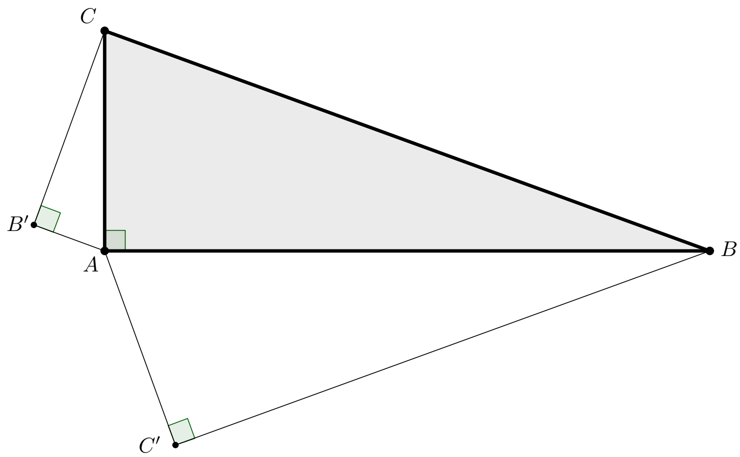
and so we conclude $2^S = 2^{\left(\frac{1}{1006}\right)} = \boxed{\frac{1}{1006}.}$ □

Solution 2. For any $b > 0$, we have $2 = b^{\log_b(2)}$, and then raising both sides to the power $\frac{1}{\log_b(2)}$ gives the identity $2^{\frac{1}{\log_b(2)}} = b$. Applying this to each term in the sum, we see

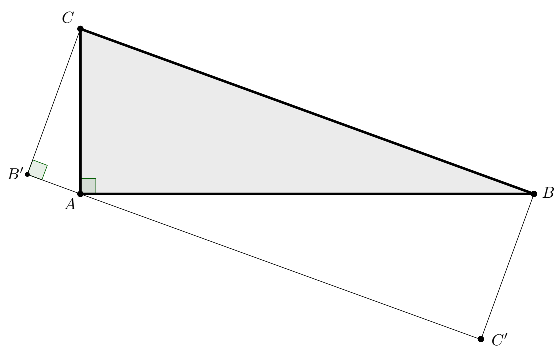
$$\begin{aligned} 2^S &= 2^{\frac{1}{\log_{2/3}(2)} + \frac{1}{\log_{3/4}(2)} + \cdots + \frac{1}{\log_{2011/2012}(2)}} \\ &= 2^{\frac{1}{\log_{2/3}(2)}} 2^{\frac{1}{\log_{3/4}(2)}} \times \cdots \times 2^{\frac{1}{\log_{2011/2012}(2)}} \\ &= \frac{2}{\cancel{3}} \cdot \frac{\cancel{3}}{\cancel{4}} \cdot \frac{\cancel{4}}{\cancel{5}} \cdots \frac{\cancel{2011}}{\cancel{2011}} \cdot \frac{\cancel{2011}}{2012} = \frac{2}{2012} = \boxed{\frac{1}{1006}.} \end{aligned}$$

□

Problem 4. Let $\angle BAC = 90°$. Let B' and C' be points such that triangles $\triangle C'BA$ and $\triangle B'CA$ are similar to $\triangle ABC$. If $\triangle ABC$ has area 13 and $\triangle B'CA$ has area 5, find the area of $\triangle C'BA$.



Solution 1. If we flip the bottom triangle horizontally, we arrive at the following rectangle (noting that $\angle CAB'$ and $\angle BAC'$ are complementary):



Now the three apparent triangles have equal altitudes. Since the sum of the bases of the two smaller triangles equals the base of the large triangle, that same relationship holds also between their areas, and so

$$\text{Area}(\triangle C'BA) = \text{Area}(\triangle B'CA) - \text{Area}(\triangle ABC) = 13 - 5 = \boxed{8.}$$

□

Solution 2. Let a, b, and c be the sides opposite A, B, and C, respectively. By the Pythagorean Theorem, we have $b^2 + c^2 = a^2$. Since the area of each triangle is a common multiple, say k, of the square of its hypotenuse, we have

$$\text{Area}(\triangle C'BA) = kc^2 = k(a^2 - b^2) = ka^2 - kb^2 = \text{Area}(\triangle B'CA) - \text{Area}(\triangle ABC) = 13 - 5 = \boxed{8.}$$

□