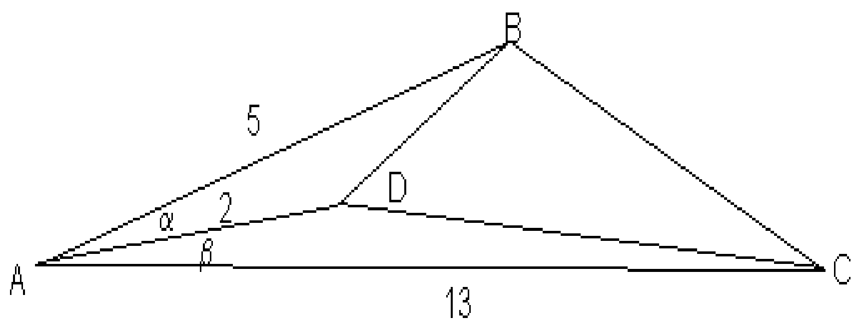
Leapfrog 2001: Solutions

1. D is a point inside triangle ABC, which has an acute angle at A. A lies 5 units from B, 13 from C, and 2 from D. Triangle ABD has area 3, and triangle ADC has area 5. What is the area of triangle BCD?

Solution: The area of triangle ABD is $(1/2) \cdot 5 \cdot 2\sin(\alpha) = 3$, so $\sin\alpha = 3/5$. The area of triangle ACD is $(1/2) \cdot 13 \cdot 2\sin(\beta) = 5$, so $\sin\beta = 5/13$. Therefore the area of triangle ABC is $(1/2) \cdot 5 \cdot 13\sin(\alpha+\beta) = (65/2)(\sin\alpha\cos\beta + \cos\alpha\sin\beta) = (65/2)((3/5)(12/13) + (4/5)(5/13)) = 28$. The area of triangle BCD is $28 - 3 - 5 = 20$.



2. For what real number k does the system of linear equations

$$\begin{array}{ccccccc} x & + & ky & + & 2kz & = & 0 \\ x & + & 2y & + & 3z & = & 0 \\ x & + & 3y & + & 2z & = & 0 \end{array}$$

have a solution where x, y, z are not all zero?

Solution: Subtracting the second and third equations, we learn that $y = z$, so $x = -5y$. Substituting these facts into the first equation yields $(-5+3k)y = 0$, so either $y = 0$ (which implies $x = y = z = 0$) or $k = 5/3$.

3. Evaluate and simplify

$$\frac{1}{1+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{4}} + \frac{1}{\sqrt{4}+\sqrt{5}} + \cdots + \frac{1}{\sqrt{99}+\sqrt{100}}.$$

Solution: The trick is to note that $1/(\sqrt{n} + \sqrt{n+1}) = \sqrt{n+1} - \sqrt{n}$, so

$$\frac{1}{1+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{4}} + \frac{1}{\sqrt{4}+\sqrt{5}} + \cdots + \frac{1}{\sqrt{99}+\sqrt{100}} =$$

$$(\sqrt{2}-1) + (\sqrt{3}-\sqrt{2}) + (\sqrt{4}-\sqrt{3}) + (\sqrt{5}-\sqrt{4}) + \cdots + (\sqrt{100}-\sqrt{99}) = 9.$$

4. Quadrilateral $ABCD$ has sides $AB = 13$, $BC = 11$, $CD = 8$ and $DA = 4$. What is the greatest possible area for this quadrilateral?

Solution: Since we are given the lengths of AB and BD, the area of triangle ABD, which is $AB \cdot AD \cdot \sin(\angle A)/2$, is greatest when $\angle A = 90^o$. Likewise, the area of triangle CBD is greatest when $\angle C = 90^o$. Now $AB^2 + DA^2 = 185 = BC^2 + CD^2$, so by the Pythagorean theorem, $\angle A$ and $\angle C$ can both be right angles! The area of $ABCD$ would then be equal to the sum of the areas of triangle ABD ($AB \cdot AD/2 = 26$) and triangle CBD ($CB \cdot CD/2 = 44$), a total area of 70.

5. Set $a_1 = 3$ and $a_{n+1} = 2a_n - 1$ for every positive integer n. Evaluate

$$a_{100} - a_1 - a_2 - \cdots - a_{99}.$$

Solution: Let $b_n = a_n - a_1 - a_2 - \cdots - a_{n-1}$. Then $b_1 = 3$, and $b_{n+1} = (2a_n - 1) - a_1 - a_2 - \cdots - a_n = b_n - 1$. Therefore $b_{100} = b_1 - 99 = -96$.

6. We can 'derive' the equation $\frac{16}{64} = \frac{1}{4}$ by cancellation: $\frac{16}{64} = \frac{1\not{6}}{\not{6}4} = \frac{1}{4}$. Of course this cancellation is invalid, but the conclusion, $\frac{16}{64} = \frac{1}{4}$, is true! Another example of the same kind of invalid cancellation leading to a true equation is $\frac{19}{95} = \frac{1\not{9}}{\not{9}5} = \frac{1}{5}$.

There is exactly one more example of this kind – that is, starting with the quotient of two *different* two-digit numbers where the one's digit of the numerator is equal to the ten's digit of the denominator, you cancel these equal digits to obtain the same fraction, but *in lowest terms.* What fraction must you start with?

Solution: The equation to solve is $\frac{10a + b}{10b + c} = \frac{a}{c}$, which after cross-multiplying and simplifying, reduces to $10ab = c(9a + b)$ (equation *). Since we require that a/c be in lowest terms, a and c have no common factors, so (*) implies that a divides 9a+b, so a divides b.

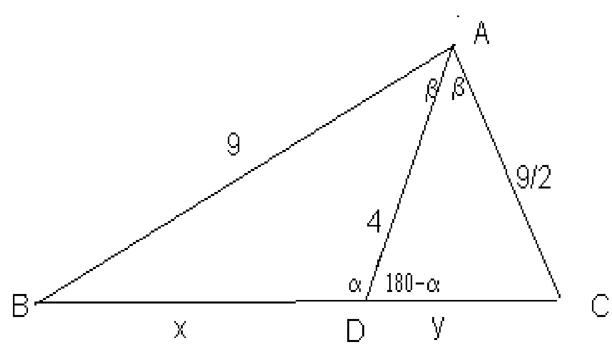
Also 10 divides $9a + b$ only when $a = b$, which cannot be, so (*) implies that c is either even, or 5.

Suppose c is even. Then $9a + b = 10a + (b - a)$ must be a multiple of 5, so $b - a$ must be a multiple of 5. The only pair of digits (a, b) where a divides b and $b - a$ is a muliple of 5 is $a = 1$, $b = 6$. Solving (*) for c, we see that

$c = 4$, and we obtain the equation $\frac{16}{64} = \frac{1}{4}$. That's not the one we're looking for.

So c must be 5. Now (*) reduces to $2ab = 9a + b$. Solving for a we find that $a = \frac{1}{2}\left(1 + \frac{9}{2b-9}\right)$. Therefore $2b - 9$ must be 1,3, or 9, so $b = 5, 6$, or 9, and the corresponding $a = 5, 2$ or 1. But $a = b = c = 5$ is impossible; the other two possibilities yield the equations $\frac{26}{65} = \frac{2}{5}$ and $\frac{19}{95} = \frac{1}{5}$. Of course, $\frac{26}{65}$ is our desired answer.

7. In triangle ABC, point D lies on side BC, AD bisects angle $\angle BAC$, $AB = 9$, $AC = 9/2$, and $AD = 4$. What is the length of side BC?



Solution: The law of sines gives $\frac{\sin\beta}{x} = \frac{\sin\alpha}{9}$ and $\frac{\sin\beta}{y} = \frac{\sin(180^o - \alpha)}{9/2}$. Since $\sin(180^o - \alpha) = \sin\alpha$, this simplifies to $x = 2y$.

The law of cosines gives $81 = 16 + x^2 - 8x\cos\alpha = 16 + 4y^2 - 16y\cos\alpha$ and $81/4 = 16 + y^2 + 8y\cos\alpha$. Therefore

$65 = 4y^2 - 16y\cos\alpha$ and $17/2 = 2y^2 + 16y\cos\alpha$.

Add these equations: $147/2 = 6y^2$, so $y = 7/2$, and so $BC = x+y = 21/2$.

8. Captain Queeg has accused Maryk, Keith, Harding, and Fred of stealing strawberries. When questioned, each makes a statement.

Maryk:	If Harding is guilty, then so is Keith.
Fred:	If Harding is innocent, then Keith is guilty.
Keith:	If Fred is guilty, then so is Maryk.
Harding:	I am innocent.

Each guilty person has lied, but no innocent person has. Name all of the actual culprits.

Solution: First, suppose Keith is guilty. Then his statement is false, so Fred must be guilty. Then Fred's statement must be false, so Keith must be innocent, a contradiction.

Therefore Keith is innocent. Maryk's and Fred's statements can't both be false, and since Keith's statement is true, Fred's guilt would imply Maryk's guilt; therefore Fred must be innocent. Since Fred's statement is true and Keith is innocent, Harding must be guilty. Therefore Maryk's statement is false. The culprits are Maryk and Harding.