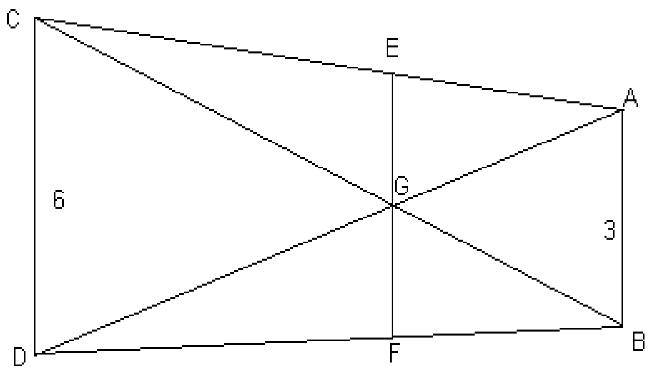
Math Field Day 2002

Leapfrog 2002 – Solutions

1. In the figure below, lines AB, CD and EF are all parallel. If AB has length 3, and CD has length 6, find the length of EF.



By similar triangles, $\frac{EG}{AB} = \frac{CE}{CA}$ and $\frac{EG}{CD} = \frac{AE}{CA}$. Adding these equations yields $\frac{EG}{AB} + \frac{EG}{CD} = \frac{EG}{3} + \frac{EG}{6} = 1$, so $EG = 2$. By symmetry, $FG = 2$, as well, so $EF = 4$.

2. Suppose that $f(x) = (100!)^{g(x)}$, where

$$g(x) = \frac{100}{\frac{1}{\log_2 x} + \frac{1}{\log_3 x} + \frac{1}{\log_4 x} + \frac{1}{\log_5 x} + \cdots + \frac{1}{\log_{100} x}}.$$

Express $f(x)$ in the simplest possible form.

Solution:

Since $\frac{1}{\log_2 x} + \frac{1}{\log_3 x} + \cdots + \frac{1}{\log_{100} x} = \frac{\ln 2}{\ln x} + \frac{\ln 3}{\ln x} + \cdots + \frac{\ln 100}{\ln x} = \frac{\ln(100!)}{\ln x} = \frac{1}{\log_{100!} x}$, $\quad g(x) = 100 \log_{100!} x = \log_{100!} x^{100}$, so $f(x) = x^{100}$.

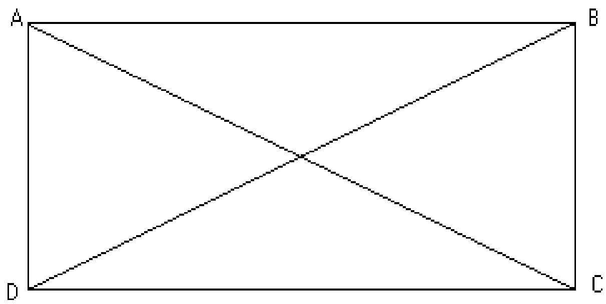
3. Evaluate this ratio of infinite sums as a fraction in lowest terms:

$$\frac{1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \frac{1}{6^3} + \cdots}{1 - \frac{1}{2^3} + \frac{1}{3^3} - \frac{1}{4^3} + \frac{1}{5^3} - \frac{1}{6^3} + \cdots}.$$

Solution: Call the numerator A and the denominator B. Then $A = B + 2(\frac{1}{2^3} + \frac{1}{4^3} + \frac{1}{6^3} + \frac{1}{8^3} + \cdots) = B + \frac{1}{4}(1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots) = B + \frac{A}{4}$.

Therefore $B = 3A/4$, so $A/B = 4/3$.

4. Each of the six line segments AB, AC, AD, BC, BD and CD in the figure below is to be colored either red or green. How many ways are there to do this so that each of the four triangles ABC, ABD, ACD and BCD has at least one red side and one green side?



Solution: First note that the segments AB, AC, AD cannot all be the same color, because in order to avoid a monochromatic triangle we would have to color all sides the triangle BCD the opposite color, which is not allowed. Therefore there are 6 possible ways to color the segments AB, AC, AD; 3 ways with 2 greens and a red, and 3 ways with 2 reds and a green.

Suppose, for example, we color AB and AD green, and AC red. Then BD must be red, and the only restriction on BC and CD is that they cannot both be red. Thus there are 3 ways to color BD, BC and CD.

For each of the 6 ways to color AB, AC, and AD, there are 3 ways to color the remaining segments. Therefore there are 18 ways to color the figure.

5. If $a_1 = a_2 = 0$, simplify the expression

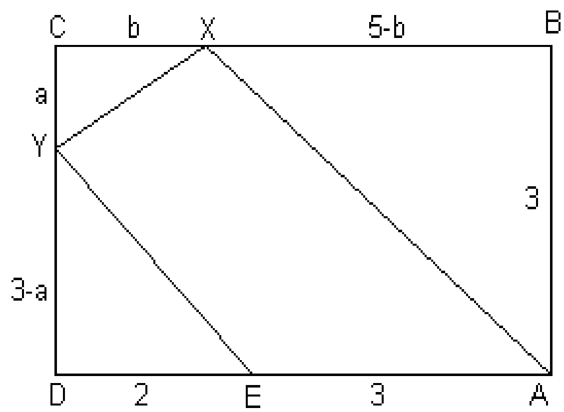
$$\sum_{k=1}^{\ell} \sum_{j=1}^{k} (a_{j+2} - 2a_{j+1} + a_j).$$

Solution: Using the "telescoping sum " $\sum_{m=1}^{n} a_{m+1} - a_m = a_{n+1} - a_1$ twice,

$$\begin{aligned} \sum_{k=1}^{\ell} \sum_{j=1}^{k} a_{j+2} - 2a_{j+1} + a_j &= \sum_{k=1}^{\ell} \sum_{j=1}^{k} (a_{j+2} - a_{j+1}) - (a_{j+1} - a_j) \\ &= \sum_{k=1}^{\ell} a_{k+2} - a_{k+1} \\ &= a_{\ell+2} - a_2 = a_{\ell+2}. \end{aligned}$$

6. In rectangle $ABCD$, the length of AB is 3, and the length of BC is 5. Sides BC and CD are mirrors. You want to shine a beam of light from vertex A so that it reflects off of sides BC at point X, reflects off of side CD at Y, and finally hits side AD at point E, which is 3 units from A. How far is X from C? (Hint: The laws of reflection require that $\angle AXB = \angle YXC$ and $\angle XYC = \angle EYD$.)

Solution:



Solution: Label the figure as above. The triangles ABX, YCX and YDE are all similar, so $\dfrac{5-b}{3} = \dfrac{b}{a} = \dfrac{2}{3-a}$. The first equation yields $a = \dfrac{3b}{5-b}$. Solving the second equation for a yields $a = \dfrac{3b}{b+2}$, so $5 - b = b + 2$, $b = 3/2$. (The reader is invited to try to find a purely geometric proof of the fact that $CX = AE/2$ regardless of the lengths of the various sides.)

7. What is the remainder when x^{100} is divided by $(x-1)^2$?

Solution: Since the divisor is a quadratic polynomial, we can write $x^{100} = p(x) \cdot (x-1)^2 + ax + b$. Set $x = 1$; we see that $1 = a + b$.

Therefore $x^{100} = p(x) \cdot (x-1)^2 + ax - a + 1 = p(x) \cdot (x-1)^2 + a(x-1) + 1$.

Now divide by $x - 1$;

$$\frac{x^{100}-1}{x-1} = (x-1) \cdot p(x) + a.$$

Since $x^{100} - 1 = (x-1)(x^{99} + x^{98} + \cdots + x + 1)$, we have

$x^{99} + x^{98} + \cdots + x + 1 = (x - 1) \cdot p(x) + a.$

Now let $x = 1$, and obtain $100 = a$ Therefore the remainder is $ax + b = 100x - 99$.

8. On a multiple choice test, one of the questions was illegible. The choice of answers was:

(a) All of the below.
(b) None of the below.
(c) All of the above.
(d) Exactly one of the above.
(e) None of the above.
(f) None of the above.

What answers are correct?

Solution: (a) cannot be true, because if it is then (e) is true and that makes (a) false. Therefore (c) is false. (b) cannot be true, because (b) says that (d) is false, and (d) would be true. Now since (a), (b) and (c) are false, (d) is false. Therefore (e) is true, which makes (f) false. The only true statement is (e).