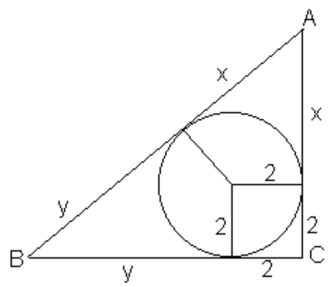
Leapfrog Solutions 2003

1. ABC is a right triangle with hypotenuse 10 and inscribed circle of radius 2. Evaluate the sum of the lengths of the two legs.

Solution: From the diagram, we see that $AB = x+y = 10$, so $AC+BC = 4+x+y = 14$.



2. Evaluate the product $\sin\frac{\pi}{48}\cdot\cos\frac{\pi}{48}\cdot\cos\frac{\pi}{24}\cdot\cos\frac{\pi}{12}$.

Solution: Repeatedly using the identity $\sin x\cos x = \frac{1}{2}\sin(2x)$, we have $\left(\sin\frac{\pi}{48}\cos\frac{\pi}{48}\right)\cos\frac{\pi}{24}\cos\frac{\pi}{12} = \frac{1}{2}\left(\sin\frac{\pi}{24}\cos\frac{\pi}{24}\right)\cos\frac{\pi}{12} = \frac{1}{4}\sin\frac{\pi}{12}\cos\frac{\pi}{12} = \frac{1}{8}\sin\frac{\pi}{6} = \frac{1}{16}$.

3. How many subsets of the set $A = \{1,2,3,4,5,6,7,8,9,10\}$ contain at least one pair of numbers x, y such that $x+y = 11$.

Solution: First count how many subsets contain no such pair. There are 5 unordered pairs of numbers x, y with $x+y = 11$, and such a subset may contain neither x nor y, x but not y, or y but not x, three possibilities. Therefore there are $3^5 = 243$ subsets of this type. There are a total of $2^{10} = 1024$ subsets of A, so the required answer is $1024 - 243 = 781$.

4. Suppose that w, x, y, z are all greater than or equal to 0, and that $6w+4x+3y+2z = 1$. What is the greatest possible value of $7w+6x+5y+3z$?

Solution: $7w+6x+5y+3z = (7w+6x+5y+3z) - \frac{5}{3}(6w+4x+3y+2z-1) = -3w - \frac{2}{3}x - \frac{1}{3}z + \frac{5}{3}$. Clearly it is best to take $w = x = z = 0$, which leaves $y = \frac{1}{3}$, and $7w+6x+5y+3z = \frac{5}{3}$.

5. The point P, inside rectangle $ABCD$, is at distance 6,2, and 7 from vertices A, B, and C, respectively. Find the distance from P to D.

Solution: Let w, x, y, z be the lengths of the perpendiculars to the sides of the square, as shown at the top of the next page. Then $PD^2 = w^2+z^2 = (w^2+x^2) - (x^2+y^2) + (y^2+z^2) = 36 - 4 + 49 = 81$, so $PD = 9$.

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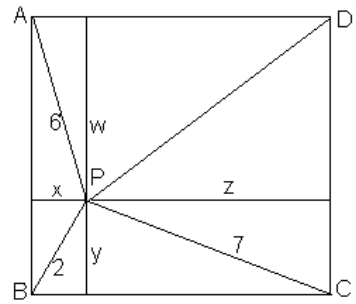

6. Larry is 5 times as old as Moe was when Curly was twice as old as Moe will be when Larry is one year younger than Curly is now. At present, everyone's age is a whole number, Larry's age is a perfect square, and his elder brother Curly's age is a perfect cube. How old is Moe?

Solution: Let L be Larry's present age, and so on. Larry will be 1 year younger than Curly is now in $C-L-1$ years, when Moe will be $M+C-L-1$. Curly is $2(M+C-L-1)$ in $2M+C-2L-2$ years, when Moe is $3M+C-2L-2$, so $L = 15M + 5C - 10L - 10$. Now L is a perfect square and a multiple of 5, so Larry must be 25 or 100. Solving this last equation for C, we get $C = -3M + 2 + \frac{11}{5}L$, so either $C = -3M + 57$ (if $L = 25$) or $C = -3M + 222$ (if $L = 100$). In either case, C is a multiple of 3 and a perfect cube. The smallest perfect cube which is a multiple of 3 is 27, and the next largest is 216, so $C = 27$. Thus either $M = 10$ or $M = 65$. Since Curly is older than Moe, $M = 10$.

7. A function f whose domain and range are both the set of positive integers, has the properties

$$f(1) = 1, \quad f(2n) = f(n) \quad \text{and} \quad f(2n+1) = f(2n) + 1.$$

Evaluate $f(2003)$.

Solution: A little experimentation shows that $f(n)$ is the number of 1's in the binary representation of n. Since $2003 = 11111010011_2$, $f(2003) = 8$.

8. There are four suspects in a robbery, Pinky, Stuffy, Rusty and The Professor. Each makes a statement.

Pinky says: The Professor is guilty.

Stuffy says: Pinky is innocent.

Rusty says: Either Pinky or Stuffy is innocent.

The Professor says: If Rusty is guilty, then so is Pinky.

Guilty people always lie, innocent people never do. Name all of the culprits.

Solution: If Rusty is guilty, his statement must be false, so Pinky is guilty. Thus the Professor's statement is true, and he is innocent. Therefore Pinky is lying and so guilty, Stuffy is lying and so guilty, and Rusty is lying. The culprits are Pinky, Stuffy and Rusty.