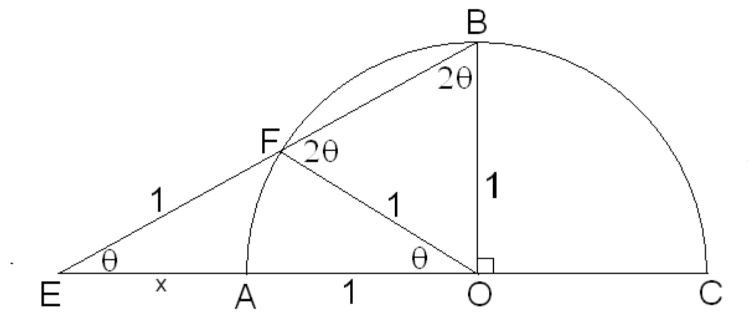
Leap Frog 2005 Solutions

1. In the figure below, AC is a diameter of the circle with center O and radius 1. Point B lies on the circle, $\angle AOB = 90^o$, point E lies outside the circle, line segment EB cuts the circle at point F, and EF has length 1. What is the length of line segment EA?



Solution: Let x be the length of AE, and let $\angle OEF = \theta$. Then $\triangle FEO$ is isosceles, so $\angle EOF = \theta$, and $\angle BFO = 2\theta$ since it is exterior to $\triangle FEO$. Also $\triangle OBF$ is isosceles, so $\angle FBO = 2\theta$, as well.

Looking at right triangle $\triangle BOE$, we have $3\theta = 90^o$, so $\theta = 30^o$, and $\tan\theta = \tan 30^o = 1/\sqrt{3} = 1/(x+1)$, so $x = \sqrt{3} - 1$.

2. Suppose p is a polynomial. When p is divided by $x-1$, the remainder is 3. When p is divided by $x-3$, the remainder is 1. What is the remainder when p is divided by $(x-1)(x-3)$?

Solution: We know that $p(x) = q(x)(x-1)(x-3) + (ax+b)$. Therefore $p(1) = a+b = 3$ and $p(3) = 3a+b = 1$. The solution to the system $a+b=3, 3a+b=1$ is $a=-1, b=4$. The remainder is $r(x) = -x+4$.

3. Express the sum $\frac{1}{3}+\frac{1}{15}+\frac{1}{35}+\frac{1}{63}+\frac{1}{99}+\frac{1}{143}+\cdots+\frac{1}{9999}$ as a fraction in lowest terms.

Solution: The problem asks for the sum

$$\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\frac{1}{7\cdot 9}+\frac{1}{9\cdot 11}+\frac{1}{11\cdot 13}+\cdots+\frac{1}{99\cdot 101}.$$

Let's compute the first few partial sums:

$$\frac{1}{1\cdot 3}=\frac{1}{3}.$$

$$\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}=\frac{2}{5}.$$

$$\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}=\frac{3}{7}.$$

$$\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\frac{1}{7\cdot 9}=\frac{4}{9}.$$

The pattern seems clear: $\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\cdots+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}$.

(One way to prove this is to note that
$\frac{1}{(2n-1)(2n+1)}=\frac{1}{2}\left(\frac{1}{2n-1}-\frac{1}{2n+1}\right)$, so $\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\cdots+\frac{1}{(2n-1)(2n+1)}=$
$\frac{1}{2}\left(\left(\frac{1}{1}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{7}\right)\cdots+\left(\frac{1}{2n-1}-\frac{1}{2n+1}\right)\right)=\frac{n}{2n+1}$.)

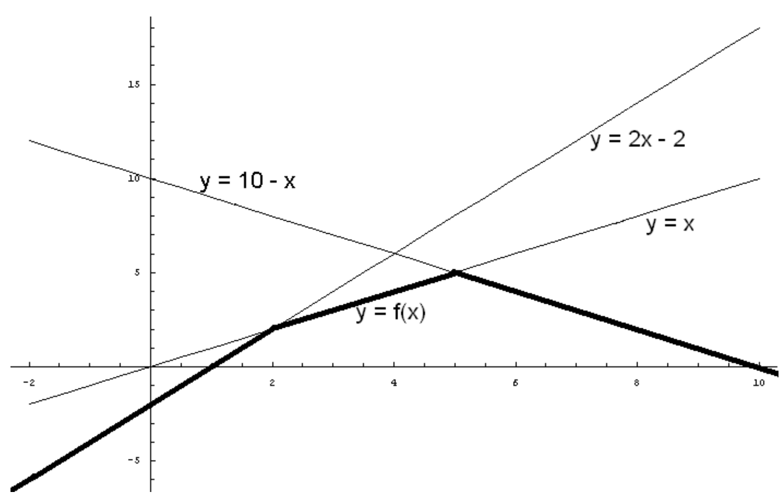
Therefore $\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\frac{1}{7\cdot 9}+\frac{1}{9\cdot 11}+\frac{1}{11\cdot 13}+\cdots+\frac{1}{99\cdot 101}=\frac{50}{101}$.

4. Twice as many men as women enter a round-robin badminton tournament, yet the ratio of wins by women to wins by men is 7:5. How many women entered the tournament? (A round-robin tournament is one in which every player plays every other player exactly once. There are no ties in badminton.)

Solution: Say n women and $2n$ men enter. Then there will be $\frac{n(n-1)}{2}$ woman-woman matches and $2n^2$ woman-man matches, for a total of $\frac{n(5n-1)}{2}$ matches in which women play. There are a total of $\frac{3n(3n-1)}{2}$ matches, so we must have $\frac{n(5n-1)}{2}\geq\frac{7}{12}\cdot\frac{3n(3n-1)}{2}$, which simplifies to $n\leq 3$. If only 1 woman enters, there are 3 contestants and only 3 matches; she cannot win 7/12 of 3 matches. Likewise, if 2 women enter, there will be 15 matches, and again 7/12 of 15 is not a whole number. The number of women who enter must be 3.

5. For each x, define $f(x)$ to be the smallest of x, $2x-2$, and $10-x$. What is the the maximum value taken by $f(x)$ over all real numbers x?

Solution: The functions $y=x$, $y=2x-2$ and $y=10-x$ are graphed below. The dark line is the graph of $f(x)$, the minimum of these three functions. It is clear that the maximum value of $f(x)$ occurs where $x=10-x$, i.e. $x=5$, and so the maximum value of $f(x)$ is $f(5)=5$.



6. Evaluate the infinite sum $\sum_{n=1}^{\infty} \frac{n}{2^n} = \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \frac{5}{32} + \cdots$.

Solution: If you read the rectangular sum below down the columns, you can see that it is the sum required by the problem. If you read across the columns, you can see that these are all just geometric series!

$$\begin{pmatrix} \frac{1}{2} & + & \frac{1}{4} & + & \frac{1}{8} & + & \frac{1}{16} & + & \frac{1}{32} & + & \frac{1}{64} & + & \cdots \\ & & \frac{1}{4} & + & \frac{1}{8} & + & \frac{1}{16} & + & \frac{1}{32} & + & \frac{1}{64} & + & \cdots \\ & & & + & \frac{1}{8} & + & \frac{1}{16} & + & \frac{1}{32} & + & \frac{1}{64} & + & \cdots \\ & & & & & + & \frac{1}{16} & + & \frac{1}{32} & + & \frac{1}{64} & + & \cdots \\ & & & & & & & + & \cdots & & & & \end{pmatrix} = \begin{pmatrix} 1 \\ +\frac{1}{2} \\ +\frac{1}{4} \\ +\frac{1}{8} \\ +\cdots \end{pmatrix} = 2.$$

7. Each edge of a regular tetrahedron has length one. What is the shortest distance between points on two non-adjacent edges?
(Hint: One way to construct a regular tetrahedron is to start with a cube, and choose 4 vertices no two of which share an edge of the cube.)

Solution: This tetrahedron can be constructed by taking 4 nonadjacent vertices of a cube of side $1/\sqrt{2}$. Non-adjacent edges of the tetrahedron occur on opposite faces of the cube, so the line connecting the center of two such faces is perpendicular to both edges. The distance is $1/\sqrt{2}$.

8. There are four suspects in a crime. Each makes a statement.

Mrs. Rittenhouse: Exactly one of us is guilty.

Mrs. Teasdale: Two of us did it.

Mrs. Claypool Three of us conspired to commit the crime.

Mrs. Upjohn: All four of us were involved.

Assume that the guilty suspects are lying, the innocent ones are not. Name all of the guilty suspects.

Solution: The statements contradict one another, so at least 3 of them are false. They can't all be false, for if they were the suspects would all be guilty, and Mrs. Upjohn would be telling the truth. Therefore the only truth-teller is Mrs. Claypool. The guilty suspects are the other three, Mrs. Rittenhouse, Mrs. Teasdale and Mrs. Upjohn.