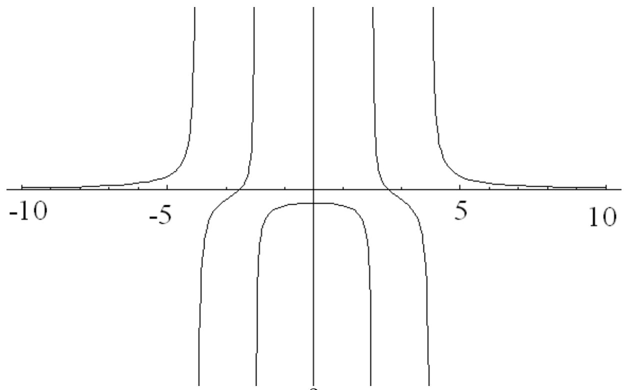
Leap Frog 2006 solutions

1. Suppose that $e^x - e^{-x} = \sqrt{5}$. Evaluate $e^x + e^{-x}$.

Solution: $(e^x + e^{-x})^2 = e^{2x} + 2 + e^{-2x} = 4 + (e^{2x} - 2 + e^{-2x}) = 4 + (e^x - e^{-x})^2 = 4 + 5 = 9$, so $e^x + e^{-x} = 3$.

2. How many real numbers x satisfy the equation $\dfrac{x^2}{x^2-4} + \dfrac{x^2}{x^2-16} = 4$?



Solution: $\dfrac{x^2}{x^2-4} - 1 = \dfrac{4}{x^2-4}$ and $\dfrac{x^2}{x^2-16} - 1 = \dfrac{16}{x^2-16}$, so the given equation is equivalent to $\dfrac{4}{x^2-4} + \dfrac{16}{x^2-16} = 2$.

The graph of the $y = \dfrac{4}{x^2-4} + \dfrac{16}{x^2-16}$ has vertical asymptotes $x = \pm 2$ and $x = \pm 4$. Since $y = x^2 - 4$ and $y = x^2 - 16$ are increasing for $x > 0$, $y = \dfrac{4}{x^2-4}$ and $y = \dfrac{16}{x^2-16}$ are decreasing for $0 < x < 2$, $2 < x < 4$ and $x > 4$. Since the graph of $y = \dfrac{4}{x^2-4} + \dfrac{16}{x^2-16}$ is symmetric about the y-axis, it must look like the graph above. The given equation has 4 solutions.

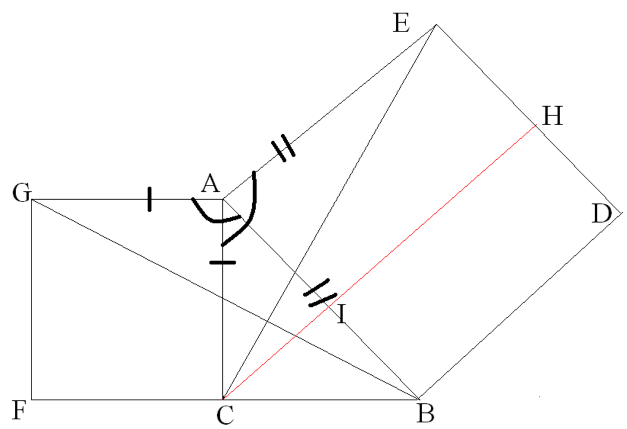
3. Suppose that a, b, c, d are real numbers, and $a^2 + b^2 + c^2 + d^2 = 1$. What is the greatest possible value of $ab + ac + ad + bc + bd + cd$?

Solution:

$$\begin{aligned} 0 &\leq (a-b)^2 + (a-c)^2 + (a-d)^2 + (b-c)^2 + (b-d)^2 + (c-d)^2 \\ &= 3(a^2 + b^2 + c^2 + d^2) - 2(ab + ac + ad + bc + bd + cd) \\ &= 3 - 2(ab + ac + ad + bc + bd + cd), \end{aligned}$$

so $ab + ac + ad + bc + bd + cd \leq 3/2$, with equality when $a = b = c = d = 1/2$.

4. You are given a right triangle ABC with right angle at C, and squares $ABDE$ and $ACFG$ constructed outwardly on two sides of the triangle. If $AC = 15$ and $BC = 17$, compute the ratio $\dfrac{\text{area of square } ACFG}{\text{area of triangle } ACE}$.



Solution: Draw in BG. Note that square $ACFG$ and triangle GAB have the same base (AG) and the same height (AC), so the area of $ACFG$ is twice the area of AGB.

Next look at triangles AGB and ACE; they are congruent (side-angle-side), so they have the same area. Therefore the area of $ACFG$ is twice the area of ACE. The required ratio is 2.

Aside: We have just reproduced the main step in Euclid's beautiful proof of the Pythagorean Theorem. Drop a perpendicular CH from C to DE. Triangle ACE also has the same base (EA) and height (AI) as the rectangle $AIHE$. It follows that $ACFG$ and $AIHE$ have the same area. By a symmetric argument, the square on the side BC has the same area as rectangle $BDHI$. Therefore, IH splits the square on the hypothenuse AB into two rectangles, one of which has the same area as the square on leg AC, and the other has the same area as the square on leg AB.

5. Factor completely the expression $\quad a(b-c)^3 + b(c-a)^3 + c(a-b)^3$.

Solution: Note that when $a = b$ or $b = c$ or $c = a$, the given expression is 0. Therefore $(a-b)(b-c)(c-a)$ is a factor. Furthermore, every term in $a(b-c)^3 + b(c-a)^3 + c(a-b)^3$ is of degree 4, while every term in $(a-b)(b-c)(c-a)$ is of degree 3, so the factorization must be

$$a(b-c)^3 + b(c-a)^3 + c(a-b)^3 = (a-b)(b-c)(c-a)(Aa + Bb + Cc)$$

for some real numbers A, B, C. Let $a = 0, b = 1$; we obtain

$$c^3 - c = (-1)(1-c)c(B + Cc) = Cc^3 + (B-C)c^2 - Bc$$

for all c; it follows that $B = C = 1$. By symmetry, $A = 1$, as well. The required factorization is

$$a(b-c)^3 + b(c-a)^3 + c(a-b)^3 = (a-b)(b-c)(c-a)(a+b+c).$$

6. The number 9 can be written as the sum of two or more consecutive positive integers in exactly 2 ways, namely $4+5$ and $2+3+4$. In how many ways can the number 42 be written as the sum of two or more consecutive positive integers?

Solution: First suppose we want to write the number X as the sum of an odd number of consecutive positive integers. Then we have

$$X = (a-b) + \cdots + (a-1) + a + (a+1) + \cdots + (a+b) = a(2b+1)$$

so X is the product of a positive integer a and an odd positive integer $2b+1$ with $a > b$.

To write X as the sum of an even number of consecutive positive integers, we must have

$$X = (a-(b-1)) + \cdots + (a-1) + a + (a+1) + \cdots + (a+b) = a(2b) + b = (2a+1)b$$

so X is the product of a positive integer b and the odd positive integer $2a+1$, with $a \geq b$.

This shows that number of ways to write X as the sum of an odd number of consecutive positive integers is equal to the number of odd factors of X which are greater than 1. The number 42 has 3 such factors, 3,7,21, so the answer is 3. In fact, $42 = 13+14+15 = 3+4+5+6+7+8+9 = 9+10+11+12$.

7. Suppose x and y are real numbers. What is the greatest possible real-number value of the expression $x\sqrt{1-y^2} + y\sqrt{1-x^2}$?

Solution: Since $|x| \leq 1$ and $|y| \leq 1$ (otherwise $x\sqrt{1-y^2}+y\sqrt{1-x^2}$ is not a real number), we may set $x = \sin\alpha$ and $y = \sin\beta$, where α and β are in the first or fourth quadrant. Then $x\sqrt{1-y^2} + y\sqrt{1-x^2} = \sin\alpha\cos\beta + \sin\beta\cos\alpha = \sin(\alpha+\beta)$. The greatest possible value is 1, obtained when, for example, $x = 1, y = 0$.

8. There are 5 suspects in a robbery: Homer, Marge, Bart, Lisa, and Maggie. Each makes a statement.

Homer:	Lisa is guilty.
Marge:	Homer and Bart are not both innocent.
Bart:	If Homer is guilty, then so is Marge.
Lisa:	If Marge is guilty, then so is Maggie.
Maggie:	If Lisa is guilty, then so is Bart.

Each innocent suspect told the truth, each guilty suspect lied. Name all of the culprits.

Solution: Maggie's statement says that Lisa is innocent or Bart is guilty, that is, either Homer is guilty or Bart is guilty. Therefore Maggie's statement is equivalent to Marge's statement, which means that Lisa is innocent. Therefore Homer is guilty, so Marge is innocent, so Bart is guilty and Maggie is innocent. The culprits are Homer and Bart.