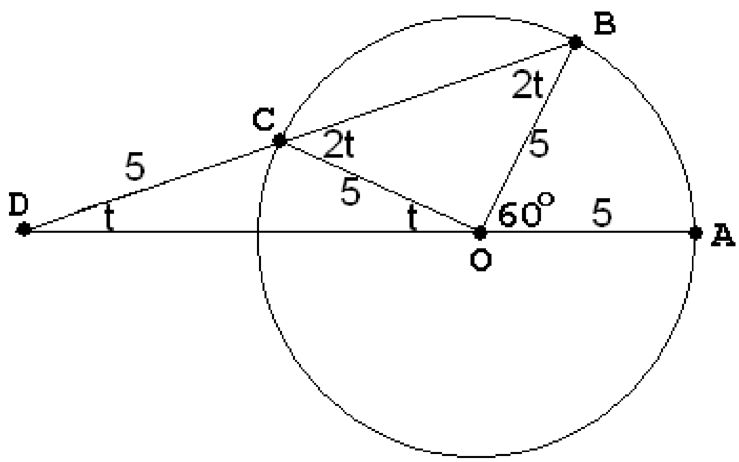
Leapfrog 2008

1. In the diagram, points A, B, C lie on the circle with radius 5 and center O, line BC meets line AO at D, $CD = 5$, and $\angle AOB = 60^o$. Evaluate angle BDA.



Solution: Let $t = \angle BDA$ and $u = \angle BOC$. Since $CD = CO$, $\angle COD = t$, so $\angle BCO = 2t$ and $\angle CBO = 2t$. Now $4t + u = 180^o$ and $t + u + 60^o = 180^o$, and these two equations easily imply $t = 20^o$.

2. Three positive integers have a product of 7200 and a sum of 75. Find the largest of the three integers.

Solution: Call the three integers x, y, z. Since $7200 = 2^5 \cdot 3^2 \cdot 5^2$, x, y, z must contain 3 as a factor twice and 5 as a factor twice. Since $x+y+z = 75$ is a multiple of 3 and 5 both, the two 3's and the two 5's must occur in the same integer; say $x = 9a$ and $y = 25b$. That leaves five 2's to distribute. Since $x+y+z$ is odd, two of our integers must be even and one odd; the 2's must be either distributed $2 \cdot 16$ or $4 \cdot 8$. A bit of trial and error shows that the only possibility is that $x = 9$, $y = 50$, $z = 16$. The largest of our integers is 50.

3. What is the smallest possible value of

$$f(x) = |\sin x + \cos x| + |\tan x + \cot x| + |\sec x + \csc x| \ ?$$

Solution: Note that $|\tan x + \cot x| = \left|\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}\right| = \frac{1}{|\sin x \cos x|} =$

$\dfrac{2}{|\sin 2x|} \geq 2$, with equality when $x = \frac{\pi}{4} + \frac{k\pi}{2}$. When $x = \frac{3\pi}{4}$, $\sin x + \cos x = \sec x + \csc x = 0$, and $\tan x + \cot x = -2$; each term in f is minimized. Then minimum value is $f(\frac{3\pi}{4}) = 2$.

4. Suppose 3 coins are biased so that they fall heads with probabilities 1/3, 1/5 and 1/7, respectively. Toss all 3 coins; what is the probability you obtain an odd number of heads?

Solution: P(odd number of heads)=P(HHH or HTT or THT or TTH) = $\dfrac{1}{3} \cdot \dfrac{1}{5} \cdot \dfrac{1}{7} + \dfrac{1}{3} \cdot \dfrac{4}{5} \cdot \dfrac{6}{7} + \dfrac{2}{3} \cdot \dfrac{1}{5} \cdot \dfrac{6}{7} + \dfrac{2}{3} \cdot \dfrac{4}{5} \cdot \dfrac{1}{7} = \dfrac{3}{7}$

5. Suppose x, y, z are positive integers such that

(i) $x < y < z$,

(ii) z has no factors larger than 1 in common with xy, and

(iii) $(x+y)(x+z)(y+z)$ is a multiple of xyz.

Find z.

Solution: We are told that $z|(x+y)(x+z)(y+z) = (x+y)(xy+xz+yz+z^2)$, so $z|xy(x+y)$. Hypothesis (ii) implies $z|x+y$. Say $kz = x+y$; then $kz = x+y < 2z$, so $k = 1$. Therefore $z = x+y$.

Now (iii) reduces to $xy|(x+z)(y+z) = xy + (x+y)z + z^2 = xy + 2z^2$. Therefore $xy|2z^2$, so (ii) implies that $xy|2$, that is, $x = 1$, $y = 2$, $z = 3$.

6. Suppose that when you multiply out the polynomial

$$P(x) = (x+1)(x+2)(x+3)\cdots(x+2007)(x+2008),$$

you obtain

$$P(x) = a_{2008}x^{2008} + a_{2007}x^{2007} + a_{2006}x^{2006} + a_{2005}x^{2005} + \cdots + a_3x^3 + a_2x^2 + a_1x + a_0.$$

Evaluate

$$\frac{a_0 + a_2 + a_4 + \cdots + a_{2004} + a_{2006} + a_{2008}}{a_0 + a_1 + a_2 + \cdots + a_{2006} + a_{2007} + a_{2008}}.$$

Solution: The denominator is $P(1)$. The numerator is $\frac{1}{2}P(1) + \frac{1}{2}P(-1)$, which equals $\frac{1}{2}P(1)$, since $(x+1)$ is a factor. The answer is 1/2.

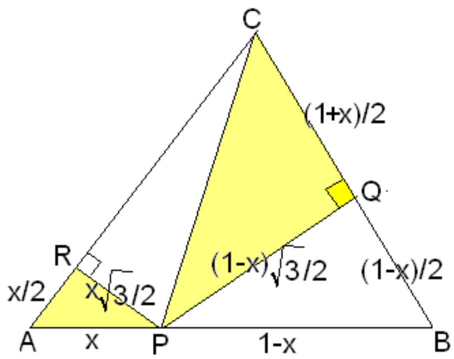
7. ABC is an equilateral triangle of side length 1. Points P, Q, R lie on sides AB, BC, CA, respectively, PQ is perpendicular to BC and PR is perpendicular to CA. If AP has length x, find the total shaded area.

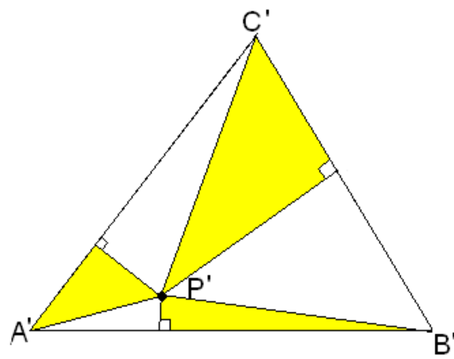
Solution: Since ΔPAR is a $30^o - 60^o - 90^o$ right triangle, $RA = x/2$ and $RP = x\sqrt{3}/2$, so the area of $\Delta PAR = x^2\sqrt{3}/8$.

Similarly, ΔPBQ is a $30^o - 60^o - 90^o$ right triangle, so $PQ = (1-x)\sqrt{3}/2$ and $BQ = (1-x)/2$, so $CQ = 1 - (1-x)/2 = (1+x)/2$. Therefore the area of $\Delta PBQ = (1-x^2)\sqrt{3}/8$.

Therefore, the total shaded area is $\sqrt{3}/8$. It is independent of x!

For a more challenging exercise, prove that the shaded area in $\Delta A'B'C'$ is also independent of the position of P' inside triangle $A'B'C'$





8. Three suspects have been arrested for a heinous crime, Bob Rowland, Horatio Jamison, and Frank Wagstaff. Each makes two statements.

Bob Rowland:	I am not guilty. The culprit is bald.
Horatio Jamison:	Rowland is not guilty. The culprit has long hair.
Frank Wagstaff:	I am not guilty. Jamison did it.

No suspect lied twice. Exactly one of the suspects is guilty. Which one?

Solution: Wagstaff cannot be the culprit, because if he is then both of his statements are false. Rowland cannot be guilty, because if he is, then Jamison's first statement is false, so his second statement is true, so Rowland's second statement is false, and of course his first statement is false. The culprit is Jamison.