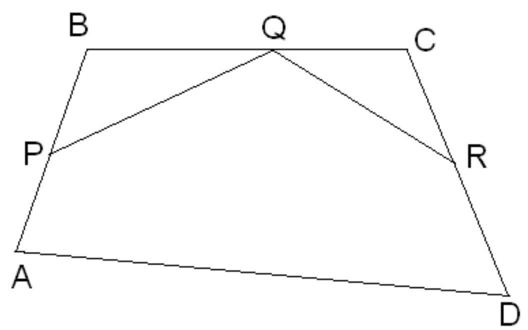
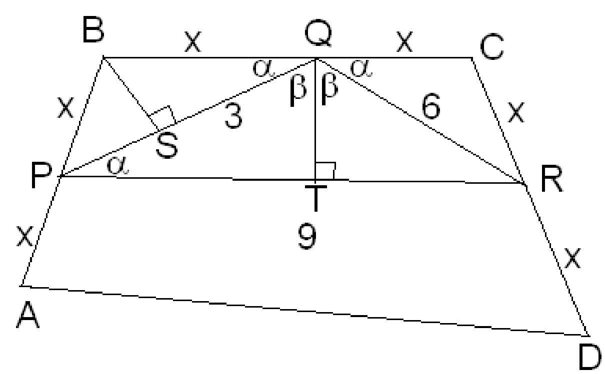
Leap Frog 2009

1. Suppose $ABCD$ is a quadrilateral, and sides AB, BC, CD are of equal length and have midpoints P, Q, R, respectively. PQR is an isosceles triangle, with $PQ = QR = 6$ and $PR = 9$. Find the common length of AB, BC, CD.



Solution:



Since $\triangle BPQ \cong \triangle CQR$ (side-side-side), $\angle BQP = \angle CQR = \alpha$. Drop perpendicular QT from Q to PR. Since triangle PRQ is isosceles, $\angle PQT = \angle RQT$, and so $\angle BQT = \angle CQT$. Therefore BC is parallel to PR, and so $\angle TPQ = \alpha$, as well. Now drop perpendicular BS from B to PQ; since triangle PBQ is isosceles, $SQ = \frac{1}{2}PQ = 3$. Furthermore, right triangles BSQ and QTP and similar, so $\frac{x}{3} = \frac{6}{9/2}$, so $x = 4$. Therefore $AB = BC = CD = 8$.

2. At a certain pet store there are the same number of dogs, cats and anteaters. However, half of the dogs think they are cats, two thirds of the cats think they are anteaters, and three fourths of the anteaters think they are dogs. (The rest think correctly.) If I choose at random an animal that thinks it is a cat, what is the probability that it is a really a dog?

Solution: Suppose that there are 36 animals. Then there are 12 dogs, and 6 of them think they are cats. There are 12 cats, and 4 of them think they are cats. None of the anteaters think they are cats, so there are a total of 10 animals that think they are cats, and 6 of them are dogs. The probability is 6/10 or 3/5.

3. Find the value of x that minimizes the sum

$$\left(\ln\left(\frac{2}{x}\right)\right)^2 + \left(\ln\left(\frac{5}{x}\right)\right)^2 + \left(\ln\left(\frac{8}{x}\right)\right)^2 + \left(\ln\left(\frac{25}{x}\right)\right)^2 + \left(\ln\left(\frac{50}{x}\right)\right)^2.$$

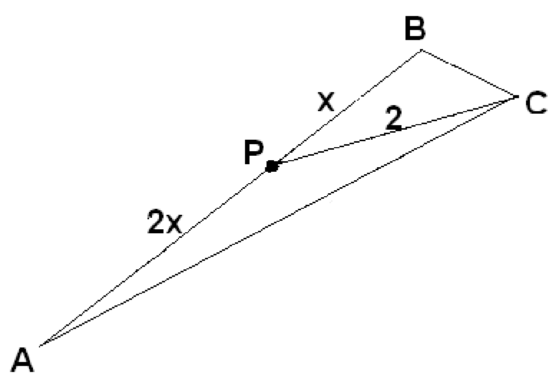
Solution: $\left(\ln\left(\frac{2}{x}\right)\right)^2+\left(\ln\left(\frac{5}{x}\right)\right)^2+\left(\ln\left(\frac{8}{x}\right)\right)^2+\left(\ln\left(\frac{25}{x}\right)\right)^2+\left(\ln\left(\frac{50}{x}\right)\right)^2$
$=(\ln 2-\ln x)^2+(\ln 5-\ln x)^2+(\ln 8-\ln x)^2+(\ln 25-\ln x)^2+(\ln 50-\ln x)^2$
$=5(\ln x)^2-2(\ln 2+\ln 5+\ln 8+\ln 25+\ln 50)x+((\ln 2)^2+(\ln 5)^2+(\ln 8)^2+(\ln 25)^2+(\ln 50)^2)$
$=5(\ln x)^2-2(\ln 100,000)x+((\ln 2)^2+(\ln 5)^2+(\ln 8)^2+(\ln 25)^2+(\ln 50)^2)$.
is a quadratic polynomial in the variable $\ln x$. It is minimized when $\ln x$ is at the vertex, namely $\ln x=(\ln 10^5)/5=\ln 10$, so $x=10$.

4. For every finite nonempty set S, let $\Pi(S)$ be the product of its elements. Evaluate the sum of $(\Pi(S))^2$ over all nonempty subsets S of $\{1,2,3,4,5,6,7\}$ that contains no consecutive numbers.

Solution: For each n, let P_n be the value obtained if you replace $\{1,2,3,4,5,6,7\}$ by $\{1,2,\cdots,n\}$. Then $P_1=1$, $P_2=1^2+2^2=5$, $P_3=1^2+2^2+3^2+(1\cdot 3)^2=23$, so you may guess that $P_n=(n+1)!-1$.

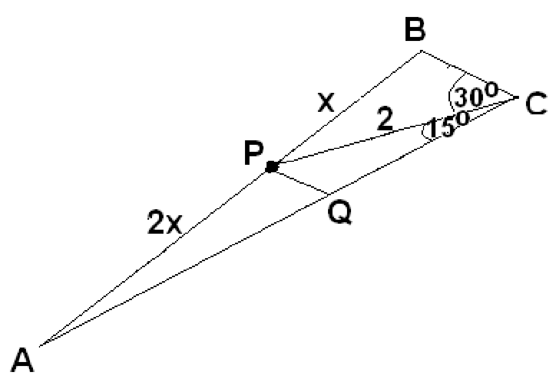
You can prove this by induction by noting that $P_n=P_{n-1}+n^2+n^2P_{n-2}=n!-1+n^2+n^2(n-1)!-n^2=(n+1)!-1$. In particular, $S_7=8!-1=40319$.

5.



Suppose that ABC is a triangle, that point P on side AB is such that $AP:PB=2:1$, PC has length 2, $\angle BCP=30^o$ and $\angle PCA=15^o$. Find the length of AC.

Solution:

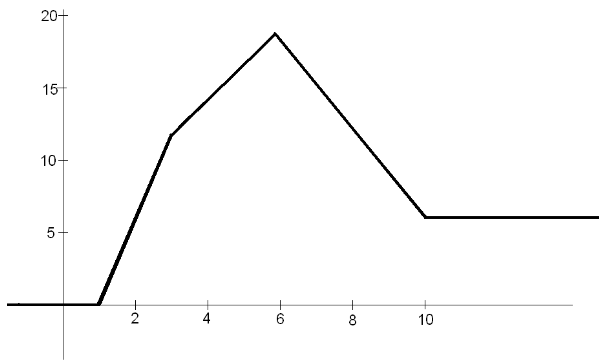


Find point Q on AC so that PQ is parallel to BC. then $AC:QC=AB:PB=3$, so $AC=3QC$. Apply the law sines to triangle PQC: $QC=2\cdot\frac{\sin 30^o}{\sin 135^o}=\sqrt{2}$. Therefore $AC=3\sqrt{2}$.

6. For each real number y, define $y^+ = \max\{y, 0\}$.
What is the greatest value taken by the function

$$f(x) = 6(x-1)^+ - 4(x-3)^+ - 5(x-6)^+ + 3(x-10)^+ \quad ?$$

Solution: The graph of f consists of line segments, connected at the points where $x = 1, 3, 6, 10$. The maximum (and minimum) value exists because for $x < 1$, the slope is 0, and for $x > 10$, the slope is $6 - 4 - 5 + 3 = 0$. The graph below shows that it peaks at $x = 6$, so the largest value f takes is $f(6) = 18$.



7. List all of the integers m for which the polynomial $p_m(x) = x^3 - mx^2 + mx - (m^2+1)$ has at least one integer root.

Solution: The equation $p(x) = 0$ may be rewritten $(x^2+m)(x-m) = 1$. Since x and m are both required to be integers, this implies that $x - m = \pm 1$.

If $x = m+1$, then $x^2 + m = m^2 + 3m + 1 = 1$, which has solutions $m = 0$ and $m = -3$.

If $x = m-1$, then $x^2 + m = m^2 - m + 1 = -1$, which has no integer solutions.

The possible values of m are 0 and -3. In fact, $p_0(1) = p_{-3}(-2) = 0$.

8. There are five suspects in the Dukenfield caper. Each makes a statement.

L.E. Whipsnade:	If Sousé and Criblecoblis are guilty, then so is McGargle.
C.J. Twillie:	If McGargle is innocent, then so is Sousé.
E. Sousé:	If McGargle is innocent, then so is Criblecoblis.
E.P. McGargle:	If Twillie and Sousé are both guilty, then so is Whipsnade.
O. Criblecoblis:	Whipsnade is innocent, but Twillie and Sousé are both guilty.

Each guily man lied, and each innocent man told the truth. Name all of the innocent men.

Solution: McGargle's statement must be true; if Twillie and Sousé are not both guilty, then it is vacuously true, and it Twillie and Sousé are both guilty, then Twillie's statement is false, which implies that McGargle is innocent. Criblecoblis's statement is the negation of McGargle's statement, so it must be false. Now that we know that McGargle is innocent and Criblecoblis is guilty, it follows that Sousé is guilty, then that Twillie is guilty, and finally that Whipsnade is guilty. The only innocent man is McGargle.