Leap Frog solutions

1. Evaluate $\sec(\tan^{-1}(\cos(\sin^{-1}(1/5))))$.
Solution: Since $\sec(\tan^{-1}(x)) = \sqrt{1+x^2}$ and $\cos(\sin^{-1}(x)) = \sqrt{1-x^2}$,
$\sec(\tan^{-1}(\cos(\sin^{-1}(1/5)))) = \sqrt{1+1-(1/5)^2} = 7/5$.

2. In how many ways can we choose positive integers in nondecreasing order so that their sum is 2010 and the largest and smallest differ by at most 1? That is, in how many ways can we choose positive integers $a_1 \le a_2 \le \cdots \le a_n$ such that $a_1 + a_2 + \cdots + a_n = 2010$ and $a_n - a_1 \le 1$?

Solution: For each number n from 1 through 2010, there is exactly one way to choose the integers $a_1, a_2, \cdots, a_n$. Divide 2010 by n, and say q is the quotient and r is the remainder, so $2010 = qn + r$. Take the first n-r integers to be q and the last r to be $q+1$.

There are 2010 ways to do this.

3. Suppose $0 \le a \le 1$, $0 \le b \le 1$, $0 \le c \le 1$ and $0 \le d \le 1$. What is the largest possible value of $S = a + b + c + d - (ab + bc + cd + da)$?

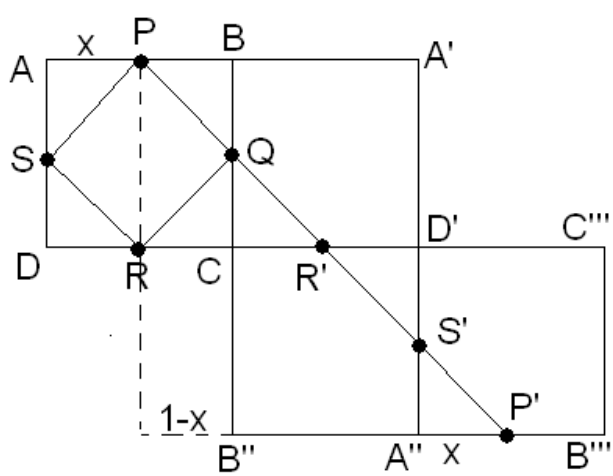
Solution: Suppose b, c, d are known, and we must choose a. Then since $S = a(1-b-d) + b + c + d - bc - cd$, we will choose $a = 1$ if $b + d \le 1$ and $a = 0$ if $b + d > 1$.

Say $a = 0$. Then $S = c + (b+d)(1-c)$. Since $1 - c \ge 0$, we will choose $b = d = 1$, and since $S = c + 2(1-c) = 2 - c$, we will choose $c = 0$, and so $S = 2$.

Now say $a = 1$. Then $S = 1 + c(1-b-d)$, so we will choose $b = d = 0$, and since $S = 1 + c$, we will choose $c = 1$, so again $S = 2$.

The largest possible value of S is 2.

4. Given a square $ABCD$ with each side of length 1, choose points P, Q, R, S on sides AB, BC, CD, DA, respectively. What is the smallest possible value of the perimeter of quadrilateral $PQRS$?



Solution: Reflect $ABCD$ across side BC to square $A'BCD'$, then reflect $A'BCD'$ across side CD' to square $A''B''CD'$, and finally reflect $A''B''CD'$ across side $A''D'$ to square $A''B'''C'''D'$. Then the perimeter of $PQRS$ is the same as the total length of $PQR'S'P'$. This is shortest when it is a straight line, and its length is $\sqrt{8} = 2\sqrt{2}$.

5. Find all pairs of numbers (x, y) with $0 < x < 1$ and $0 < y < 1$ such that $11x + 5y$ and $5x + 2y$ are both integers.

Solution: If $11x + 5y$ and $5x + 2y$ are both integers, then so is $2(11x + 5y) - 5(5x + 2y) = -3x$, so $x = \frac{1}{3}$ or $\frac{2}{3}$.

Likewise, $5(11x + 5y) - 11(5x + 2y) = 3y$ is an integer, so $y = \frac{1}{3}$ or $\frac{2}{3}$.

There are only 4 pairs (x, y) to try, and 2 of them work, $(\frac{1}{3}, \frac{2}{3})$ and $(\frac{2}{3}, \frac{1}{3})$.

6. Two players, Ann and Bill, play a game. There are 12 chips on a table. Ann picks up a_1 chips, where $1 \leq a_1 \leq 11$. Then Bill picks up a_2 chips, where $1 \leq a_2 \leq a_1$. If there are any left, Ann picks up a_3 chips, where $1 \leq a_3 \leq a_2$. The players continue alternating; on the nth turn, whosever turn it is picks up a_n chips, where $1 \leq a_n \leq a_{n-1}$. The player who takes the last chip wins.

In order to win the game, how many chips should Ann take on her first turn?

Solution: Ann should take 4 chips, leaving 8. If she takes an odd number leaving an odd number, Bill wins by taking 1. If she takes 2 leaving 10, Bill takes 2 and Bill wins, because if they continue taking 2 on each turn Bill wins, and if Ann ever takes 1 leaving an odd number, Bill also wins. If Ann takes 6 or more, Bill takes the rest and wins immediately.

If Ann takes 4 chips leaving 8, she wins because any of the choices for Bill – 1,2,3,4 – enable Ann to win for reasons similar to those in the previous paragraph.

(For a harder problem, start with 24 chips – this also has a unique answer.)

7. Find a set of numbers such that

(i) Each is a power of 3 times a power of 5 (that is, $3^m \cdot 5^n$);

(ii) none is a multiple of any other; and

(ii) they sum to 1205.

Solution: Since 1205 is not a multiple of 3, one of the numbers must be a power of 5. To be sure that it doesn't divide any of the others, choose the highest power of 5 that we can that is less than 1205. We can't choose $5^4 = 625$, because $1205 - 625 = 580$ is still not a muliple of 3, so choose $5^3 = 125$ instead.

Now $1205 - 125 = 1080 = 3^3 \cdot 5 \cdot 8$, and $8 = 3 + 5$.

Putting all of this together, we have $1205 = 5^3 + 3 \cdot 5(3 + 5) = 5^3 + 3^4 \cdot 5 + 3^3 \cdot 5^2$.

8. There are 6 suspects in a heist, Cathy A, Anne B, Jane S, Anne C, Kathy H, and Kathy P. Guilty suspects always lie; innocent suspects always tell the truth. Each makes a statement.

Cathy A.: Anne C and I are both guilty.

Anne B: Jane S told me that either she (Jane S) is guilty or I (Anne B) am guilty, but not both.

Jane S: I never said that.

Anne C: My foot itches.

Kathy H: Anne C's foot itches.

Kathy P: Anne C's foot doesn't really itch.

Solve the crime; name all of the guilty suspects.

Solution: Cathy A implicates herself, so if her statement is true she must be guilty, and if her statement is false she must be guilty. Therefore Cathy A is guilty, so her statement is

false, and so Anne C must be innocent. That makes Kathy H's statement true, so Kathy H must be innocent. That Anne B must be lying is a bit more subtle. If Anne B and Jane S are both innocent, the statement that Anne B attributes to Jane S is false, which is impossible. If Anne B is innocent and Jane S is guilty, then the statement Anne B attributes to Jane S is true, which is also impossible. Thus Anne B must be lying, which means Jane S is telling the truth and Kathy P is lying. The guilty parties are Cathy A, Anne B, and Kathy P.