Leapfrog 2011 – Solutions

1. My father is 15 years younger than my grandfather was when I was 95 years younger than my grandfather will be when my father is 50 years older than I am today. How old was my grandfather when my father was born?

Solution: Let M, F and GF be the present ages of myself, my father and my grandfather, respectively. When my father is $M+50$, my grandfather will be $(GF-F)+M+50$. When I was 95 years younger than that, $GF-F+M-45$, my grandfather was $(GF-M)+GF-F+M-45 = 2GF-F-45$. My father is 15 years younger than that, that is, we have the equation $F = 2GF-F-60$. This implies that $GF-F = 30$.

2. Four horses race. How many different ways can they finish, if ties are possible?

Solution: If there are no ties, $4! = 24$ finishes are possible.

If there is exactly one tie between 2 horses, there are $\binom{4}{2} = 6$ ways to choose the horses that tie, and $3! = 6$ ways to order the 3 groups; 36 finishes are possible.

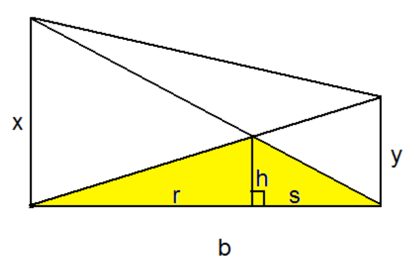
If 3 horses tie, there are 4 ways to choose the remaining horse, and 2 ways to order the groups; 8 finishes are possible.

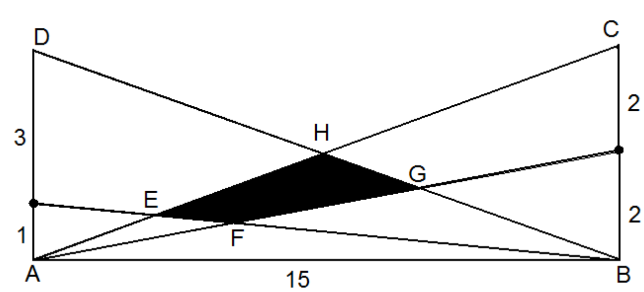
If there are 2 two-way ties, there are 3 ways to choose the groups and 2 ways to order groups; 6 finishes are possible.

Or there could be a 4-way tie.

The total number of possible finishes is $24+36+8+6+1 = 75$

3. Quadrilateral $ABCD$ has right angles at A and B. Find the area of the shaded region.





Solution: First consider the figure above on the left. Similar triangles tell us that $\frac{h}{x} = \frac{s}{b}$ and $\frac{h}{y} = \frac{r}{b}$. Add these equations and we have $h(\frac{1}{x}+\frac{1}{y}) = 1$, so the shaded area is $\frac{1}{2}bh = \frac{bxy}{2(x+y)}$.

Next, note that in the figure on the right, the shaded area is equal to Area($\triangle ABH$) - Area($\triangle ABE$) - Area($\triangle ABG$) + Area($\triangle ABF$). Using the formula from the paragraph above four times, the shaded area is equal to $15-6-10+5 = 4$.

4. Find the smallest positive integer x such that $(x+y^2)(x^2+y) = (x+y)^3$ for some positive integer y.

Solution: Note that

$(x+y^2)(x^2+y) = (x+y)^3 \iff x^3 + xy + x^2y^2 + y^3 = x^3 + 3x^2y + 3xy^2 + y^3$

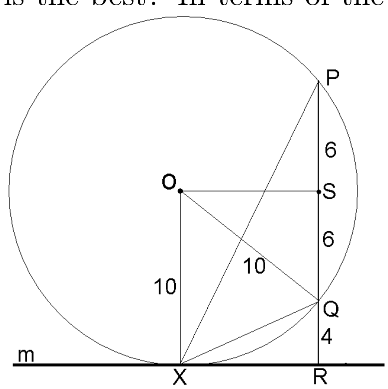
$\iff xy(xy - 3x - 3y + 1) = 0 \iff xy - 3x - 3y + 1 = 0 \iff (x-3)(y-3) = 8.$

The smallest possible value of x is $x = 4$.

5. If $x + y = 1$ and $x^3 + y^3 = \frac{1}{2}$, evaluate $x^2 + y^2$.

Solution: Let $x^2 + y^2 = A$. Then $1 = (x+y)^2 = A + 2xy$, so $xy = \frac{1-A}{2}$. Now $1 = (x+y)^3 = x^3 + y^3 + 3xy(x+y) = \frac{1}{2} + \frac{3(1-A)}{2}$, so $A = \frac{2}{3}$.

6. You are traveling along road m, looking at a billboard PQ. The billboard is perpendicular to the road. It is 12 meters across, and its nearest edge is 4 meters from the road. How far are you from the point R (the point where the extended billboard would hit the road) when your viewing angle is the best? In terms of the diagram, find the distance XR which maximizes $\angle PXQ$.



Solution: Consider the circles through P and Q which meet line m in point X; $\angle PXQ$ will be maximized when the radius is minimized, that is, when the circle is tangent to line m. Say this circle has center O, and let S be the midpoint of chord PQ. Then $QS = 6$, and the circle has radius $OQ = OX = 10$. Since ΔOSQ is a right triangle, $XR = OS = \sqrt{OQ^2 - OS^2} = 8$ meters.

7. Suppose that M is a positive integer, and there are exactly 2 positive integers n such that $4n+1$ is a divisor of $3n+M$. What is the smallest possible value of M?

Solution: Since $4n+1$ divides $3n+M$, $4n+1$ also divides $4(3n+M)=12n+4M$, and also divides $12n+4M-3(4n+1)=4M-3$. We want $4M-3$ to have exactly 2 divisors $4n+1$, so $4M-3$ should be the square of a prime number which is 1 more than a multiple of 4. The smallest such prime number is 5, so $4M-3=25$, so $M=7$.

8. There are five suspects in a serious heist, the Freak, Matt, Jon, the Mad Bum and the Beard. Each makes a statement.

The Freak:	The Mad Bum is innocent or Jon is innocent.
Matt:	The Mad Bum and Jon are both innocent.
Jon:	One of the Freak and Matt is innocent, and the other is guilty.
The Mad Bum:	The Beard and Jon are either both innocent or both guilty.
The Beard:	The Freak and the Mad Bum are both innocent

Innocent suspects never lie, guilty suspects always do. Name all of the culprits.

Solution: Suppose Matt's statement is true; then the Mad Bum and Jon are not guilty, so the Freak is telling the truth, and since Jon is telling the truth and the Freak is not guilty, Matt must be guilty. We conclude that Matt must actually be guilty.

Since Matt is guilty, Jon's statement is true if and only if the Freak is innocent. Now suppose that the Mad Bum is innocent; then the Freak's statement is true, so Jon must be innocent. Since Matt is lying, one of the Mad Bum and Jon must be guilty; we conclude that the Mad Bum must be guilty.

It follows that the Beard's statement is false. Since the Mad Bum's statement is false and the Beard is guilty, Jon must be innocent. Finally, since Jon is telling the truth and Matt is guilty, the Freak must be innocent. The culprits are Matt, the Mad Bum and the Beard.