2012 LEAPFROG PROBLEMS
WITH SOLUTIONS

Problem 1. Find the sum of the 5-digit numbers whose digits consist of one 6, one 8, one 9, and two 1's.

Solution. Consider first summing the ones digit of all of these numbers. The number 6 occurs as the ones digit of such a number once for each of the $\frac{4!}{2!} = 12$ permutations of the remaining digits, 1189, and the same for 8 and 9. The number 1 occurs in the ones spot the once for each of the $4! = 24$ permutations of the remaining digits 1689. So summing all of the ones digits of these numbers gives

$$12(6) + 12(8) + 12(9) + 24(1) = 300.$$

The argument is identical for the tens digit, hundreds digit, etc., giving a total sum of 300 ones plus 300 tens, plus 300 hundreds, etc., culminating with a grand total of:

$$300(1) + 300(10) + 300(100) + 300(1000) + 300(10000) = 300(11111) = \boxed{3333300.}$$

□

Problem 2. A relay race is held where runner A begins at the origin and runs along a line in the first quadrant until the runner hits the line $x = 1$, at which point he stops and hands the baton to runner B. Runner B runs down the line $x = 1$ until she hits the x-axis, makes a right turn, and returns to the origin along the x-axis. If the entire length of the run was 8 units, how much farther did B run than A?

Solution. Let a and b be the distances run by A and B. Since $a+b = 8$ and the racecourse is a right triangle of hypotenuse a and legs 1 and $b - 1$, the Pythagorean theorem gives

$$1 = a^2 - (b-1)^2 = (a + (b-1))(a - (b-1)) = 7(a - b + 1),$$

so $a - b + 1 = \frac{1}{7}$, and $b - a = 1 - \frac{1}{7} = \boxed{\frac{6}{7}.}$ □

Problem 3. Find the largest k such that the positive integer

$$\frac{1011(3012^n - 1) - 3011(1012^n - 1)}{1011 \times 3011}$$

ends with at least k zeros for all $n \geq 2$.

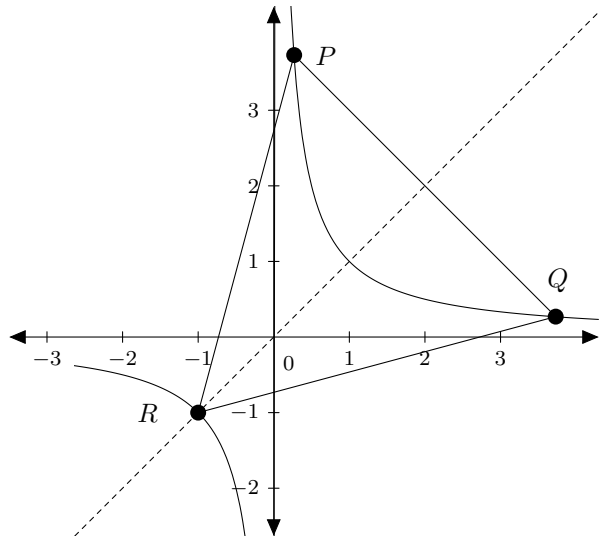
Solution. We have

$$\begin{aligned}\frac{1011(3012^n - 1) - 3011(1012^n - 1)}{1011 \times 3011} &= \frac{3012^n - 1}{3012 - 1} - \frac{1012^n - 1}{1012 - 1} \\ &= (1 + 3012 + 3012^2 + \cdots + 3012^{n-1}) - (1 + 1012 + 1012^2 + \cdots + 1012^{n-1}) \\ &= (3012 - 1012) + (3012^2 - 1012^2) + \cdots + (3012^{n-1} - 3012^{n-1}).\end{aligned}$$

Since $3012 - 1012 = 2000$ divides each term on the right, the whole expression is a multiple of 1000, and so $k \geq 3$. But when $n = 2$, the expression equals 2000 exactly, so $k \leq 3$, and we conclude $\boxed{k = 3.}$ □

Problem 4. An equilateral triangle has one vertex at $(-1,-1)$, and the other two vertices are in the first quadrant on the graph of $y = \frac{1}{x}$. Find the sum of the x-coordinates of the three vertices.

Solution. Let $R = (-1,-1)$, and label the other two vertices P and Q. Since P and Q lie on $y = \frac{1}{x}$ and are symmetric with respect to the line $y = x$, we can write $P = (x, \frac{1}{x})$ and $Q = (\frac{1}{x}, x)$ for some $x > 0$.



Since $(PR)^2 = (QR)^2$, we have

$$(x+1)^2 + \left(\frac{1}{x}+1\right)^2 = \left(x - \frac{1}{x}\right)^2 + \left(\frac{1}{x} - x\right)^2$$

Rearranging gives

$$0 = x^2 - 2x - 6 - \frac{2}{x} + \frac{1}{x^2} = \left(x + \frac{1}{x}\right)^2 - 2\left(x + \frac{1}{x}\right) - 8 = \left(x + \frac{1}{x} + 2\right)\left(x + \frac{1}{x} - 4\right)$$

Since x and $\frac{1}{x}$ are positive, the first factor cannot be zero, and hence $x + \frac{1}{x} = 4$. Now the sum of the three x-coordinates is given by

$$x + \frac{1}{x} - 1 = 4 - 1 = \boxed{3.}$$

□

Problem 5. There are unique positive integers $a < b < c$ such that

$$\frac{65}{168} = \frac{1}{a} + \frac{1}{ab} + \frac{1}{abc}.$$

Find $a + b + c$.

Solution. If $a \le 2$, the sum on the right would be greater than $\frac{1}{2}$, hence greater than $\frac{65}{168}$. If $a \ge 4$, then since $a < b < c$, we have

$$\frac{1}{a} + \frac{1}{ab} + \frac{1}{abc} \le \frac{1}{4} + \frac{1}{4\cdot 5} + \frac{1}{4\cdot 5\cdot 6} = \frac{5\cdot 6 + 6 + 1}{4\cdot 5\cdot 6} = \frac{37}{120} < \frac{1}{3} < \frac{65}{168}.$$

So we must have $a = 3$. Subtracting off the $\frac{1}{a}$ term, we get

$$\frac{3}{56} = \frac{65}{168} - \frac{1}{3} = \frac{1}{3b} + \frac{1}{3bc}.$$

Now if $b \le 6$, then

$$\frac{1}{3b} + \frac{1}{3bc} > \frac{1}{18} = \frac{3}{54} > \frac{3}{56}.$$

Similarly, if $b \ge 8$, we have

$$\frac{1}{3b} + \frac{1}{3bc} \le \frac{1}{3 \cdot 8} + \frac{1}{3 \cdot 8 \cdot 9} = \frac{10}{3 \cdot 8 \cdot 9} < \frac{3}{56}.$$

That only leaves $b = 7$, which gives

$$\frac{1}{21} + \frac{1}{21c} = \frac{3}{56},$$

or $\frac{c+1}{c} = \frac{63}{56} = \frac{9}{8}$, so $c = 8$ and $a + b + c = 3 + 7 + 8 = \boxed{18.}$ □

Problem 6. For some real value of a, the the polynomial

$$f(x) = x^3 + (3a - 5)x^2 + (a + 3)x - 4a^2 = 0$$

has the unusual property that it has three different positive real roots, each of whose reciprocal is also a root of $f(x)$. Find the largest root of $f(x)$.

Solution. Associating real numbers with their reciprocals pairs up all positive numbers except 1. Since $f(x)$ has three roots and three is odd, $x = 1$ must be a root of $f(x)$. This gives

$$0 = f(1) = 1 + (3a - 5) + (a + 3) - 4a^2 = -1 + 4a - 4a^2 = -(2a - 1)^2.$$

Thus $a = \frac{1}{2}$, and we can factor out an $(x - 1)$ from $f(x)$ to deduce the factorization:

$$f(x) = x^3 - \tfrac{7}{2}x^2 + \tfrac{7}{2}x - 1 = (x - 1)(x^2 - \tfrac{5}{2}x + 1) = (x - 1)(x - 2)(x - \tfrac{1}{2}),$$

so the largest root is $\boxed{x = 2.}$ □

Problem 7. Let a, b, c, and d be positive integers such that $n = 2^a 3^b 5^c 7^d$ has the following three properties:

- $36n$ is a fourth power of an integer;
- $\frac{5n}{3}$ is the cube of an integer;
- $\frac{n}{7}$ is an integer.

Find the least possible value of $a + b + c + d$.

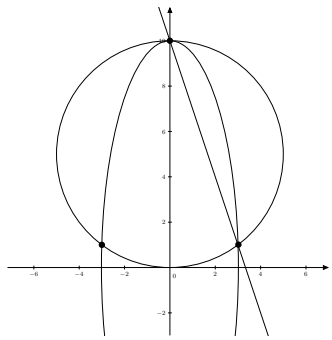
Solution. The first condition says that c and d are multiples of 4, and that a and b are each two more than a multiple of 4. The second condition says that a and d are multiples of 3, b is 1 more than a multiple of 3, and c is 2 more than a multiple of 3. The third condition says that $d \ge 1$. The least integers that meet these conditions are $a = 6$, $b = 10$, $c = 8$, and $d = 12$. This gives $a + b + c + d = \boxed{36.}$ □

Problem 8. If a and b are real numbers such that exactly one of the three equations

$$a^2 + b^2 = 10b, \qquad 11a^2 + b^2 = 100, \qquad 3a + b = 10$$

is false, find a.

Solution. Graphically, the problem is the following diagram in the ab-plane, consisting of two conics and a line: We need to find the a-coordinate of the only point on exactly two of the three curves.



We proceed algebrically to find the point: number the equations

$$a^2 + b^2 = 10b \tag{1}$$

$$11a^2 + b^2 = 100 \tag{2}$$

$$3a + b = 10. \tag{3}$$

Substituting (3) in the form $b = 10 - 3a$ into (1) and (2) respectively gives

$$a^2 + (10 - 3a)^2 = 10(10 - 3a) \qquad \text{and} \qquad 11a^2 + (10 - 3a)^2 = 100,$$

both of which simplify to give $a = 0$ or $a = 3$. Using (3) again provides the points $(a, b) = (0, 10)$ and $(3, 1)$, which in fact satisfy all *three* equations. Since the intersections points of (1) and (3) also satisfy (2), and vice versa, it must be the case that the point we're after satisfies (1) and (2), and not (3). Since we know the one intersection point $(3, 1)$ of (1) and (2), and we know that both of the conics (1) and (2) are symmetric with respect to the y-axis (note equation (1) is the circle $a^2 + (b - 5)^2 = 25$ of radius 5 centered at $(0, 5)$), a second intersection point must be $(-3, 1)$. It cannot possibly also be on the line given by (3), so we conclude $\boxed{a = -3.}$ $\square$