# 2013 LEAPFROG PROBLEMS WITH SOLUTIONS

**Problem 1.** Find the largest positive integer $n$ such that the list

$$n^2 + 1, \quad n^2 + 2, \quad \ldots, \quad n^2 + 2013$$

contains the squares of at least two positive integers.

*Solution.* We note that $(n+1)^2$ and $(n+2)^2$ are within 2013 of $n^2$ exactly when $(n+2)^2 \leq n^2 + 2013$, or equivalently, $n^2 + 4n + 4 \leq n^2 + 2013$. Subtracting $n^2 + 4$ from both sides, the solution is the largest integer $n$ such that $4n \leq 2009$, which is satisfied by $\boxed{n = 502.}$ □

**Problem 2.** Let $N = 1 \times 2 \times 3 \times \cdots \times 100$. Evaluate

$$\frac{1}{\log_2 N} + \frac{1}{\log_3 N} + \cdots + \frac{1}{\log_{100} N}$$

*Solution.* Using the identity

$$\frac{1}{\log_a b} = \frac{1}{\frac{\log a}{\log b}} = \frac{\log b}{\log a} = \log_b a,$$

we re-write the sum as

$$\log_N 2 + \log_N 3 + \cdots + \log_N 100 = \log_N(2 \times 3 \times 4 \times \cdots \times 100) = \log_N(N) = \boxed{1.}$$

□

**Problem 3.** How many digits equal 1 in the result of the following multiplication?

$$\underbrace{666\cdots6}_{\text{2013 sixes}} \times \underbrace{333\cdots3}_{\text{2013 threes}}$$

*Solution.* We have

$$\begin{aligned}
\underbrace{666\cdots6}_{\text{2013 sixes}} \times \underbrace{333\cdots3}_{\text{2013 threes}} &= 3 \times \underbrace{222\cdots2}_{\text{2013 twos}} \times \underbrace{333\cdots3}_{\text{2013 threes}} \\
&= \underbrace{222\cdots2}_{\text{2013 twos}} \times \underbrace{999\cdots9}_{\text{2013 nines}} \\
&= \underbrace{222\cdots2}_{\text{2013 twos}} \times (10^{2013} - 1) \\
&= \underbrace{222\cdots200\cdots0}_{\text{2013 twos, 2013 zeroes}} - \underbrace{222\cdots2}_{\text{2013 twos}} \\
&= 222\cdots2177\cdots78,
\end{aligned}$$

so there is only $\boxed{1}$ one in the product. □

**Problem 4.** If $a$ and $b$ are distinct numbers picked at random from the set

$$\{1, 3, 9, 27, 81, \ldots, 3^{2013}\},$$

what is the probability that the polynomial $x^2 + ax + b^2$ has at least one real root?

*Solution.* The polynomial has a real root if and only if its discriminant is non-negative, i.e., if $a^2 - 4b^2 \geq 0$. Since $a, b > 0$, this is equivalent to $a \geq 2b$. Now note that for two elements of the set above, we have $a \geq 2b \iff a > b$, so our polynomial has a real root if $a > b$ and no real root if $a < b$. By symmetry, the case $a > b$ happens exactly $\boxed{\frac{1}{2}}$ of the time. □

**Problem 5.** When expanded, the expression

$$(x+2)^{16}(x^2+1)^8(x^4-1)^4(x^8-2)^2$$

gives a polynomial of degree 64. Find the sum of its coefficients.

*Solution.* Write the expanded polynomial as $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$. We note that the value

$$f(1) = a_n 1^n + a_{n-1}1^{n-1} + \cdots + a_1(1) + a_0 = a_n + a_{n-1} + \cdots + a_1 + a_0$$

is precisely the sum of its coefficients. Since our polynomial has $x = 1$ as a root, we have $f(1) = 0$, and so the sum of its coefficients must be $\boxed{0.}$ □

**Problem 6.** Consider the circle of radius 3 whose center is the point $(6, 8)$. Find the $x$-coordinate of the point on this circle that is closest to the origin $(0, 0)$.

*Solution.* Let $P$ be the point 3 units from $(6, 8)$ on the segment with endpoints $(0, 0)$ and $(6, 8)$. Because this segment is perpendicular to the tangent of the circle at $P$, $P$ is the point of the circle nearest $(0, 0)$. Because a right triangle with legs 6 and 8 has hypotenuse 10, it is similar to a right angle with hypotenuse 3 and legs $6 \cdot \frac{3}{10} = \frac{9}{5}$ and $8 \cdot \frac{3}{10} = \frac{12}{5}$. The $x$-coordinate of $P$ is $6 - \frac{9}{5} = \boxed{\frac{21}{5}.}$ □

**Problem 7.** Let $\alpha = 3.75°$. Evaluate

$$\sin(\alpha)\cos(\alpha)\cos(2\alpha)\cos(4\alpha)\cos(8\alpha)\cos(16\alpha).$$

*Solution.* We repeatedly apply the trig identity

$$\sin(2x) = 2\sin(x)\cos(x)$$

to $x = \alpha$, $x = 2\alpha$, $x = 4\alpha$, etc., in turn to simplify our expression:

$$\begin{aligned}(\sin(\alpha)\cos(\alpha))\cos(2\alpha)\cos(4\alpha)\cos(8\alpha)\cos(16\alpha) &= \frac{1}{2}\sin(2\alpha)\cos(2\alpha)\cos(4\alpha)\cos(8\alpha)\cos(16\alpha)\\ &= \frac{1}{2}\cdot\frac{1}{2}\sin(4\alpha)\cos(4\alpha)\cos(8\alpha)\cos(16\alpha)\\ &= \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\sin(8\alpha)\cos(8\alpha)\cos(16\alpha)\\ &= \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\sin(16\alpha)\cos(16\alpha)\\ &= \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\sin(32\alpha)\\ &= \frac{1}{32}\sin(120)\\ &= \frac{1}{32}\frac{\sqrt{3}}{2} = \boxed{\frac{\sqrt{3}}{64}.}\end{aligned}$$

□

**Problem 8.** Simplify the expression

$$\frac{(x+y+z)^3 - (x^3+y^3+z^3)}{3(x+y)(x+z)}.$$

*Solution.* Because the numerator

$$(x+y+z)^3 - (x^3+y^3+z^3) \tag{1}$$

is symmetric in $x$, $y$, and $z$, compare it with

$$3(x+y)(x+z)(y+z). \tag{2}$$

Multiplying out (1) and (2) show that they are both equal to

$$6xyz + 3x^2y + 3x^2z + 3y^2x + 3y^2z + 3z^2x + 3z^2y.$$

Since (1) equals (2), dividing it by $3(x+y)(x+z)$ leaves $\boxed{y+z.}$ □