

Leapfrog 2001

1. D is a point inside triangle ABC , which has an acute angle at A . A lies 5 units from B , 13 from C , and 2 from D . Triangle ABD has area 3, and triangle ADC has area 5. What is the area of triangle BCD ?

2. For what real number k does the system of linear equations

$$\begin{aligned}x + ky + 2kz &= 0 \\x + 2y + 3z &= 0 \\x + 3y + 2z &= 0\end{aligned}$$

have a solution where x, y, z are not all zero?

3. Evaluate and simplify

$$\frac{1}{1 + \sqrt{2}} + \frac{1}{\sqrt{2} + \sqrt{3}} + \frac{1}{\sqrt{3} + \sqrt{4}} + \frac{1}{\sqrt{4} + \sqrt{5}} + \cdots + \frac{1}{\sqrt{99} + \sqrt{100}}.$$

4. Quadrilateral $ABCD$ has sides $AB = 13$, $BC = 11$, $CD = 8$ and $DA = 4$. What is the greatest possible area for this quadrilateral?

5. Set $a_1 = 3$ and $a_{n+1} = 2a_n - 1$ for every positive integer n . Evaluate

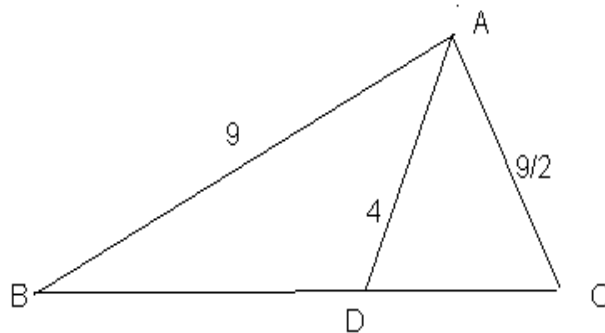
$$a_{100} - a_1 - a_2 - \cdots - a_{99}.$$

6. We can ‘derive’ the equation $\frac{16}{64} = \frac{1}{4}$ by cancellation: $\frac{16}{64} = \frac{1\cancel{6}}{\cancel{6}4} = \frac{1}{4}$.

Of course this cancellation is invalid, but the conclusion, $\frac{16}{64} = \frac{1}{4}$, is true! Another example of the same kind of invalid cancellation leading to a true equation is $\frac{19}{95} = \frac{1\cancel{9}}{\cancel{9}5} = \frac{1}{5}$.

There is exactly one more example of this kind – that is, starting with the quotient of two *different* two-digit numbers where the one’s digit of the numerator is equal to the ten’s digit of the denominator, you cancel these equal digits to obtain the same fraction, but *in lowest terms*. What fraction must you start with?

7. In triangle ABC , point D lies on side BC , AD bisects angle $\angle BAC$, $AB = 9$, $AC = 9/2$, and $AD = 4$. What is the length of side BC ?



8. Captain Queeg has accused Maryk, Keith, Harding, and Fred of stealing strawberries. When questioned, each makes a statement.

- Maryk: If Harding is guilty, then so is Keith.
- Fred: If Harding is innocent, then Keith is guilty.
- Keith: If Fred is guilty, then so is Maryk.
- Harding: I am innocent.

Each guilty person has lied, but no innocent person has. Name all of the actual culprits.