

Math Field Day 2012
Mad Hatter A

A1

Suppose that for all real numbers x and y ,

$$f(x + y) + f(x - y) = 10.$$

Then $f(y - x) = ?$

A2

Find the sum

$$1 + 2 + 4 + 5 + 7 + 8 + 10 + 11 + \dots$$

$$+49 + 50 + 52 + 53 + 55 + 56 + 58 + 59.$$

A3

Evaluate $504^2 - 502^2$.

A4

Find the sum of all of the solutions of the equation

$$||x + 4| - 3| = 2.$$

A5

Evaluate 10101^2 .

A6

If $x = 1 + \frac{1}{1 + \frac{1}{1+1}}$,

evaluate $\frac{1}{\frac{1}{x-1} - 1}$.

A7

There are 4 fnorks in a grelb.

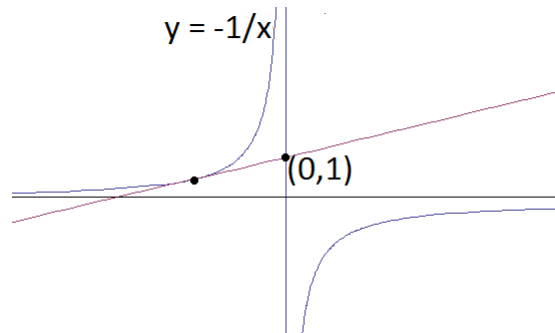
There are 11 veeblefesters is a fnork.

How many veeblefester-grelbs are there in a fnork²?

A8

If $r - \frac{1}{r} = 10$, evaluate $r^2 + \frac{1}{r^2}$.

A9

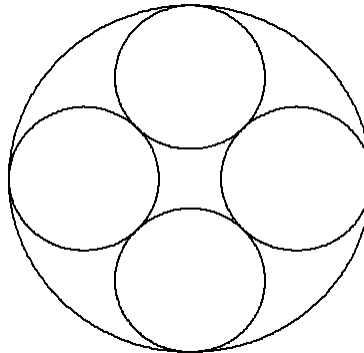


What is the greatest possible slope of a line that contains the point $(0, 1)$ and meets the hyperbola $y = -\frac{1}{x}$?

A10

If $5^{1/\log_b 5} = 2$, then $b = ?$

A11



Four small circles are inside and tangent to a larger circle, and tangent to each other. If the small circles each have radius 1, what is the radius of the large circle?

A12

Given that $\sin(2t) = \frac{1}{3}$, evaluate

$$\frac{\sin^3 t - \cos^3 t}{\sin t - \cos t}$$

A13

If r and s are the roots of the equation

$x^2 + 3x + 1 = 0$, evaluate

$$\frac{1}{r^2} + \frac{1}{s^2}.$$

A14

Find the largest value taken by the function

$$f(x) = (x + 1)^2 - 5(x - 1)(x + 1) + (x - 1)^2.$$

A15

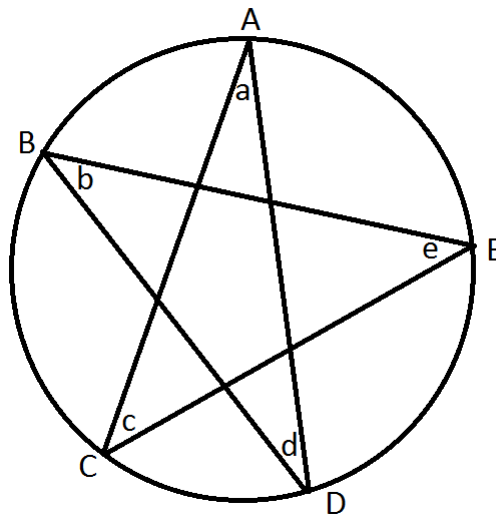
Express as a fraction in lowest terms the value of

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \cdots + \frac{1}{2011 \cdot 2012}.$$

A16

You have a collection of 100 red chips and 100 blue chips. You randomly split them into a small pile of 50 chips and a large pile of 150 chips. How many more red chips are there in the large pile than blue chips in the small pile?

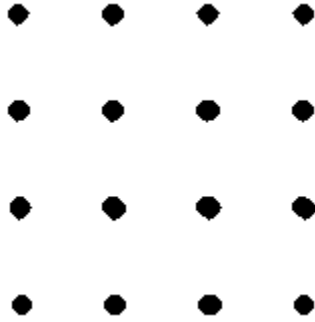
A17



The points A, B, C, D, E all lie on a circle. Find the sum of the angles

$$a + b + c + d + e.$$

A18



Each of these 16 points is 1 unit from its nearest neighbors. How many squares of area 5 have four of them as vertices?

A19

If $a_1 = 1$, $a_2 = 3$, and

$$a_n = 2a_{n-1} - a_{n-2} \quad \text{for } n > 2,$$

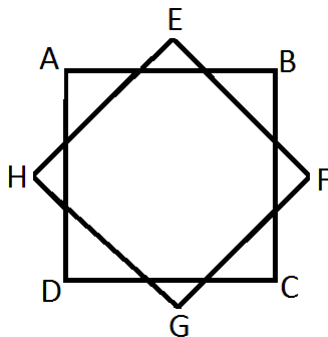
compute a_{2012} .

A20

Evaluate the infinite product

$$2 \times \sqrt{2} \times \sqrt[4]{2} \times \sqrt[8]{2} \times \sqrt[16]{2} \times \sqrt[32]{2} \times \dots$$

A21



Each side of square $ABCD$ is divided into three segments of length 1 by two sides of square $EFGH$.

What is the area of $EFGH$?

A22

I am 9 years older than my sister.

When I will be the sum of our present ages, my sister will be the age I was when she was half her present age.

How old am I?

A23

Simplify the product

$$(A - N)(B - N)(C - N) \cdots (X - N)(Y - N)(Z - N).$$

A24

Find 2^{r+1} if

$$r = \sqrt{(\log_2 3)^2 + \log_2(2/9)}.$$

A25

If $\frac{1}{x} - \frac{1}{y} = 5$,

evaluate $\frac{3x + 5xy - 3y}{x - y}$.

A26

Suppose that $a + b + c = 0$ and $abc \neq 0$. Evaluate

$$\frac{a}{b} + \frac{b}{a} + \frac{a}{c} + \frac{c}{a} + \frac{b}{c} + \frac{c}{b}.$$

A27

Suppose that

$$f(-x) = -f(x)$$

and

$$f(x + 5) = f(x)$$

for all real x .

If $f(-2) = 1$, evaluate $f(2012)$.

A28

Assume $x < 0$ and

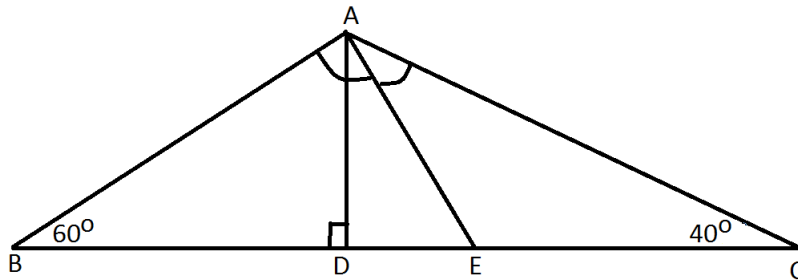
$$x^2 - 2x + 6\sqrt{x^2 - 2x + 6} = 21.$$

find x .

A29

If $|a + 1| + \sqrt{2a - b} + (a + b + c)^{2012} = 0$, find the value of $a^2 + b^2 + c^2$.

A30



In $\triangle ABC$, AE bisects $\angle BAC$,

$\angle ADB = 90^\circ$,

$\angle ABC = 60^\circ$, and

$\angle ACB = 40^\circ$.

Find $\angle DAE$.

A31

Tom wants to build a new library for three towns A , B , and C . If the distance between each two towns is 30 mi, and the new library will be the same distance from all three towns, what is the distance between the new library and town C ?

A32

Suppose

$$\log_{16} 4 - \log_3(x^3 - 189) = \frac{-5}{2}.$$

What is the value of x ?

A33

Three dice are rolled.

What is the probability that the three dice sum to 5?

A34

Suppose that when $x^k - 3x^2 + 2x - 1020$ is divided by $x - 2$, the remainder is -4 .

Find k .

A35

Suppose that $\sin 2x = \frac{2}{3}$.

Evaluate $(\sin x - \cos x)^2$.

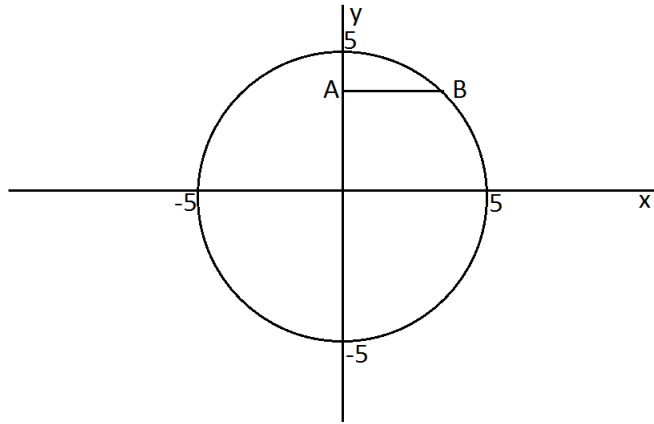
A36

Let $B(3, 2)$ be the midpoint of the line segment joining $A(-1, 1)$ and $C(m - 7, n - 3)$.
Find $m - n$.

A37

Suppose $\log_{\sqrt{2}} 8 - 3 \log_{10} 100 = \log_{1/3} y$.
Find y .

A38



Suppose A is the point $(0, 4)$, the circle in the diagram has its center at the origin, B is a point on the circle, and AB is horizontal.

Find the length of AB .

A39

Simplify the expression

$$\ln\left(1 + \frac{1}{2}\right) + \ln\left(1 + \frac{1}{3}\right) + \ln\left(1 + \frac{1}{4}\right) + \cdots + \ln\left(1 + \frac{1}{199}\right).$$

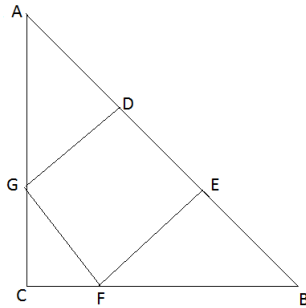
A40

Find a in degrees, if $0 < a < 90^\circ$ and

$$(1 + \sin a + \sin^2 a + \sin^3 a + \cdots) \times (1 + \cos(90^\circ + a) + \cos^2(90^\circ + a) + \cos^3(90^\circ + a) + \cdots) = 2.$$

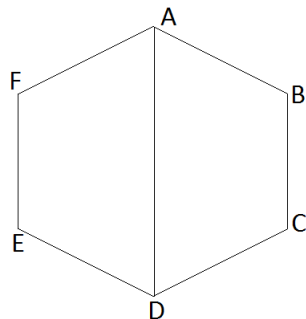
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B1



$\triangle ACB$ is an isosceles right triangle, and $DEFG$ is a square. If $AB = 12$, what is the length of EF ?

B2



Hexagon $ABCDEF$ is regular; each side has length 1. What is the length of diagonal AD ?

B3

Find the sum

$$1 + 2 + 4 + 5 + 7 + 8 + 10 + 11 + \dots \\ + 49 + 50 + 52 + 53 + 55 + 56 + 58 + 59.$$

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Evaluate $504^2 - 502^2$.

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Find the sum of all of the solutions of the equation

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B8

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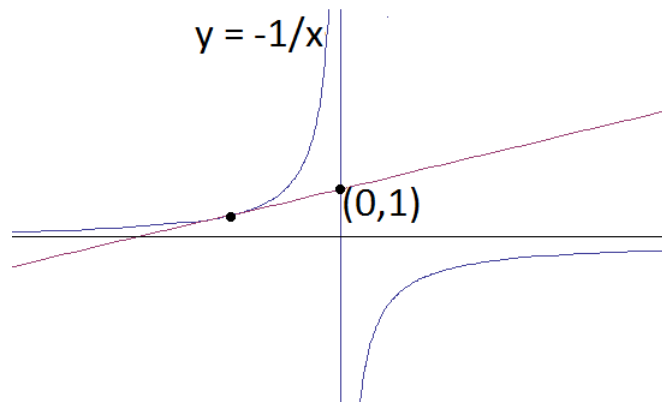
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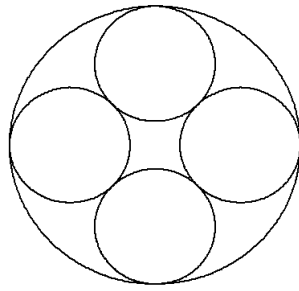
Let $A = \{1, 2, 3, 4\}$ and $B = \{4, 5, 6\}$.

Evaluate

$$(A \cup B) \cap (A \cup \overline{B}) \cap (\overline{A} \cup B),$$

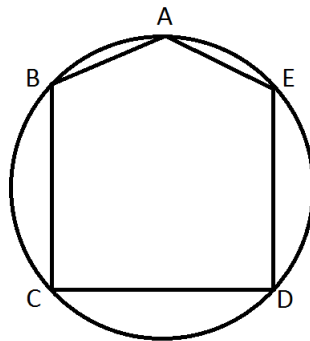
where \overline{X} is the set of all positive integers not in X .

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Four small circles are inside and tangent to a larger circle, and tangent to each other. If the small circles each have radius 1, what is the radius of the large circle?

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Pentagon $ABCDE$ is inscribed in a circle, $\angle C = \angle D = 90^\circ$, and $BC = CD = DE$. Find the measure of $\angle A$ (in degrees).

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If r and s are the roots of the equation $x^2 + 3x + 1 = 0$, evaluate

$$\frac{1}{r^2} + \frac{1}{s^2}.$$

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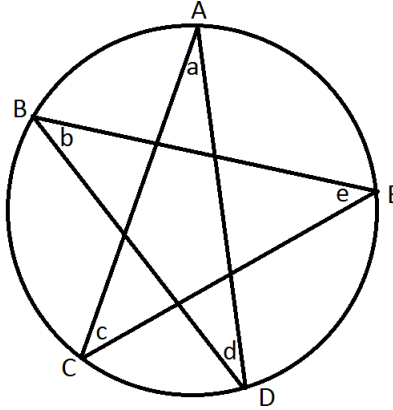
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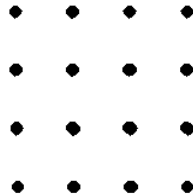
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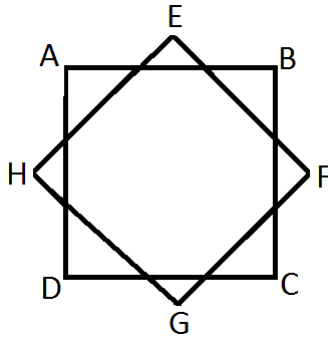
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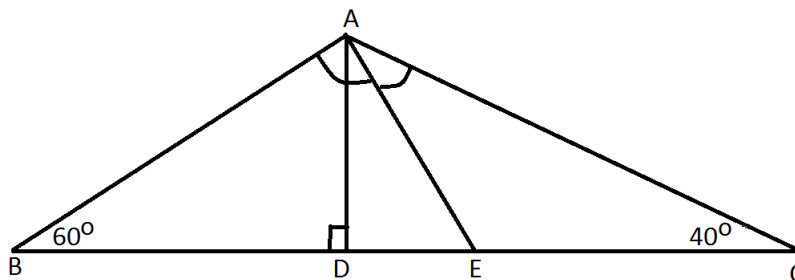
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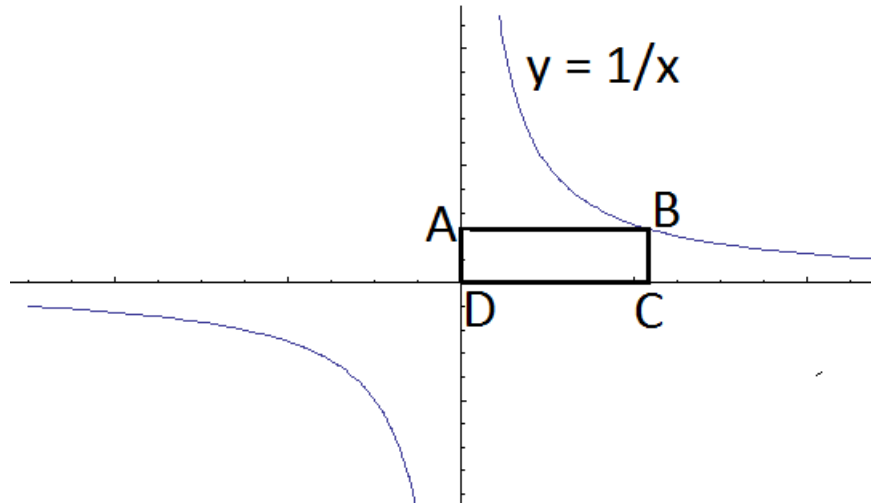
Find k .

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Suppose that $B(3, 2)$ is the midpoint of the line segment joining $A(m - 2, n + 1)$ and $C(-1, 1)$.

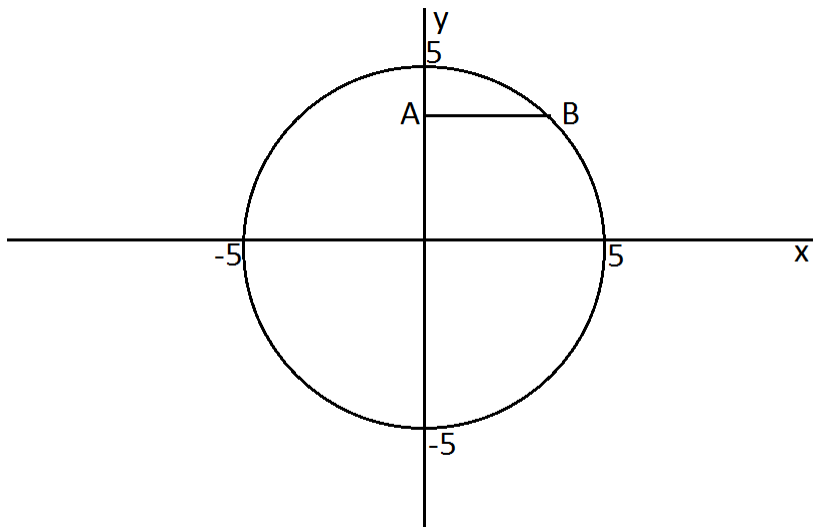
Find $m - n$.

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The curve is the graph of $y = 1/x$. Find the area of rectangle $ABCD$.

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Suppose A is the point $(0, 4)$, the circle in the diagram has its center at the origin, B is a point on the circle, and AB is horizontal.

Find the length of AB .

B37

Solve for x :

$$\left(x + \frac{1}{x}\right)^2 - \left(x - \frac{1}{x}\right)^2 = x.$$

B38

Suppose that for all real numbers x and y ,

$$f(x + y) + f(x - y) = 10.$$

Then $f(y - x) = ?$

B39

How many ways are there to arrange the letters in

PEPPER ?

B40

An integer m is perspicacious provided

If m is odd implies that m is even,
then m is even.

Which of the following is correct?

- (A) Every integer is perspicacious.
- (B) No integer is perspicacious.
- (C) Some integers are perspicacious and some are not.