Guarding Galleries

For any integer $n \geq 3$, an n-gon is an n-sided polygon consisting of n points

$$A_1, \ldots, A_n \tag{1}$$

in the plane and the n segments

$$A_1A_2,\ A_2A_3,\ \ldots,\ A_{n-1}A_n,\ A_nA_1, \tag{2}$$

provided that we never pass through a point more than once in following the segments in (2) in order from A_1 to A_n and back to A_1. We refer to the points in (1) as vertices, the segments in (2) as sides. In other words, an n-gon is a polygon with n sides that does not cross itself.

A 3-gon is a triangle (Figure 1). Figures 2a and 2b are two 4-gons, or quadrilaterals. Figure 3a is not a 4-gon and Figure 3b is not a 5-gon because they each pass through points twice.

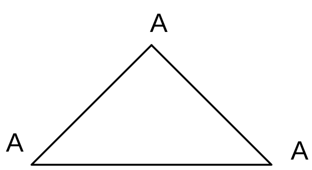


Figure 1

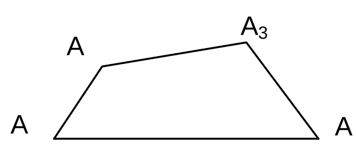


Figure 2a

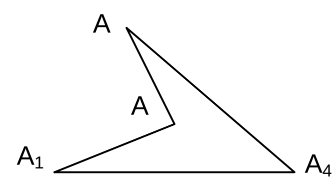


Figure 2b

A diagonal of a polygon is a segment whose endpoints are two vertices and whose other points lie inside the polygon. For example, consider the 7-gon in Figure 4 with $A_1 - A_7$ as vertices and the solid lines as sides. Segments A_1A_3, A_1A_4, A_2A_4, and A_4A_7 are diagonals, but A_1A_6 is not (because it passes outside the 7-gon), and neither is A_2A_7 (because it contains A_4, which is not inside the 7-gon).

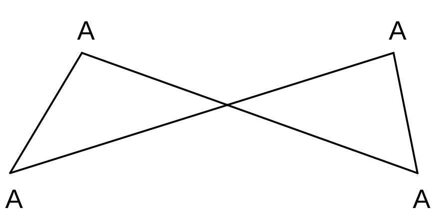


Figure 3a

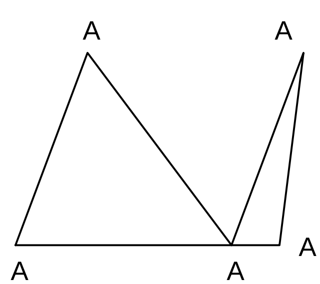


Figure 3b

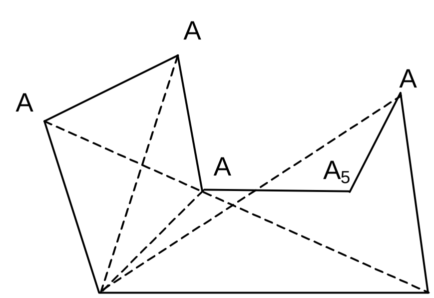


Figure 4

Tear off page 7, use it for your answers to Problems 1 through 3, and make it the first page of the answers you turn in.

<u>Problem 1.</u> Draw all the diagonals of the 8-gon on top of page 7. How many diagonals are there?

A <u>triangulation</u> of a polygon is a set of diagonals that intersect only at their endpoints and divide the interior of the polygon into triangles. For example, each of the Figures 5a and 5b is a triangulation of the polygon in Figure 4, but Figure 6 is not (since the diagonals A_1A_3 and A_2A_4 intersect).

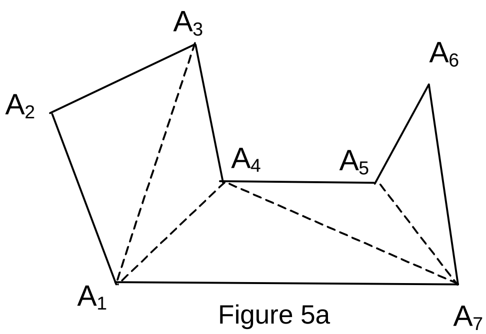


Figure 5a

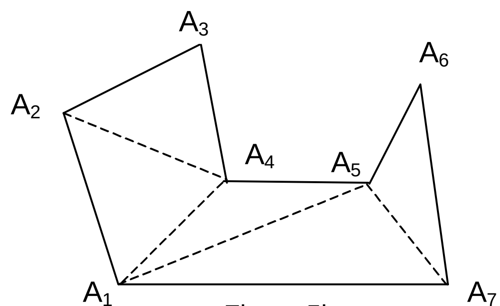


Figure 5b

<u>Problem 2.</u> Use the three figures in the middle of page 7 to draw three different triangulations of the 8-gon in Problem 1.

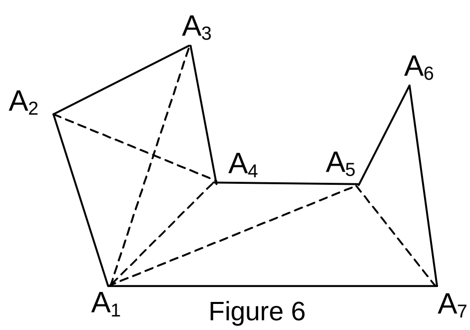


Figure 6

Let P be a triangulated polygon. Any diagonal included in the triangulation divides P into two triangulated polygons, which each have fewer vertices than P.

For example, the diagonal A_4A_7 divides the triangulated 7-gon in Figure 5a into a triangulated 5-gon and a triangulated 4-gon (Figure 7).

A triangulated polygon can be 3-colored if each vertex can be assigned one of three colors in such a way that each triangle of the triangulation has vertices of three different colors. For example, Figure 8 shows a 3-coloring of Figure 5a, where B, Y, and G represent the colors blue, yellow, and green. Of course, the choice of colors doesn't matter.

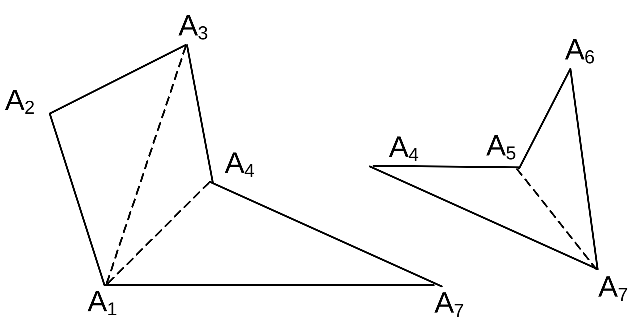


Figure 7

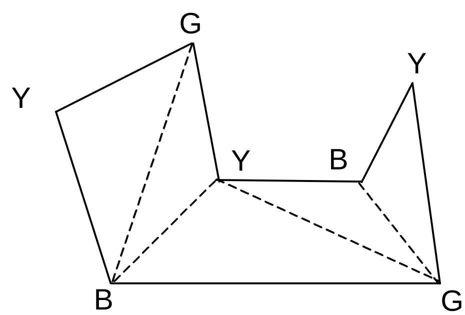


Figure 8

> Problem 3. Find a 3-coloring for the triangulated 11-gon on the bottom of page 7.

We claim that every triangulated n-gon can be 3-colored. We prove this by induction on n, that is, by working up through the possible values 3, 4, 5, ... of n. The claim obviously holds for n = 3 because the vertices of a triangle can be given three different colors.

Now consider a triangulated n-gon P for an integer $n \geq 4$. Assume that we already know that every triangulated polygon with fewer than n vertices can be 3-colored. Since $n \geq 4$, the triangulation of P includes at least one diagonal d. As noted after Problem 2, d divides P into triangulated polygons Q and R with fewer than n vertices. Q and R have 3-colorings, by the second sentence of this paragraph.

Problem 4. Show that the 3-colorings of Q and R can always be combined into a 3-coloring of P by choosing their colors appropriately.

Problem 4 completes the proof of the claim that every triangulated polygon can be 3-colored.

For any real number r, let [r] denote the greatest integer less than or equal to r. For example, [12/3], [13/3], and [14/3] all equal 4.

Problem 5. For any triangulated n-gon, prove that there is a set of at most [n/3] vertices that includes one vertex from every triangle in the triangulation. (Hint: Use the fact that the triangulation has a 3-coloring.)

For example, we have [7/3] = 2 for the triangulated 7-gon in Figure 5a. The two blue vertices in Figure 8 include one vertex from every triangle in the triangulation, and so do the two green vertices.

Problem 5 shows how many guards suffice for an art gallery. Imagine that the floor plan of the art gallery is a polygon without interior walls. Guards are to be stationed in positions so that every point in the gallery is visible from the assigned position of at least one guard.

The Art Gallery Theorem states that *a gallery formed by an n-gon needs at most [n/3] guards.* Problem 5 proves this for a gallery formed by a triangulated n-gon: guards stationed by each vertex chosen in Problem 5 can see the whole gallery because every triangle in the triangulation has a vertex with a guard. For example, the entire gallery in Figure 8 can be seen by guards standing near the two blue or two green vertices.

Problem 6. Use the preceding approach and your answer to Problem 3 to determine how to station [11/3] guards in an art gallery whose floor-plan is the 11-gon on the bottom of page 7.

The next result shows that no number of guards less than [n/3] will suffice for every n-gon.

Problem 7. For any integer n ≥ 3, show that there is an n-gon that requires [n/3] guards. Figures 9a - c suggest possibilities.

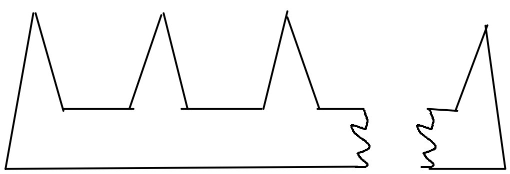


Figure 9a

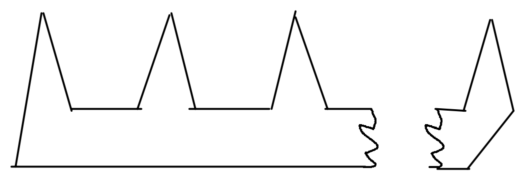


Figure 9b

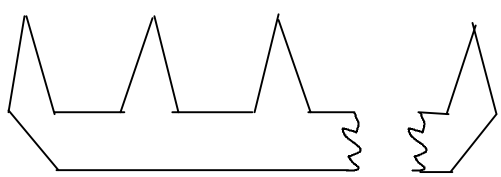


Figure 9c

We have yet to show that every polygon can be triangulated.

Problem 8. Assume that every n-gon for n ≥ 4 has a diagonal. Use this assumption to prove that every n-gon has a triangulation for n ≥ 3. Use induction on n to do the proof; that is, work up through the possible values of n.

We must still prove that every n-gon P for n ≥ 4 has a diagonal. Take a line *m* that does not intersect P, and move *m* parallel to itself until it first touches P. Then P has consecutive vertices A, B, C such that one side of *m* contains all points of P not on *m* and *m* contains B but not both A and C (Figures 10a and b). Let S be the set of all vertices of P that lie either inside triangle ABC or on segment AC between A and C.

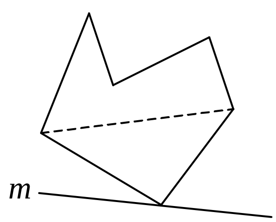


Figure 10a

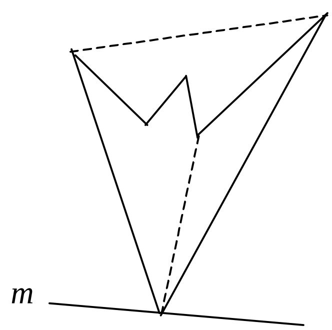


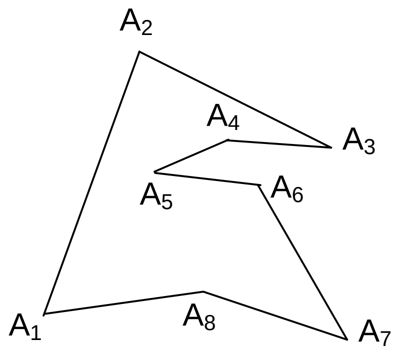
Figure 10b

> <u>Problem 9.</u> (a) Suppose that S is empty. Why is AC a diagonal of P [Figure 10a]? (b) Suppose that S is nonempty. Since it is finite, S has an element Z such that no element of S lies farther from line AC than Z does. Why is BZ a diagonal of P [Figure 10b]?

The two parts of Problem 9 show that every n-gon P for n ≥ 4 has a diagonal, as claimed. We now know that [n/3] is the smallest number of guards sufficient for every gallery given by an n-gon. This result is called the <u>Art Gallery Theorem</u>.

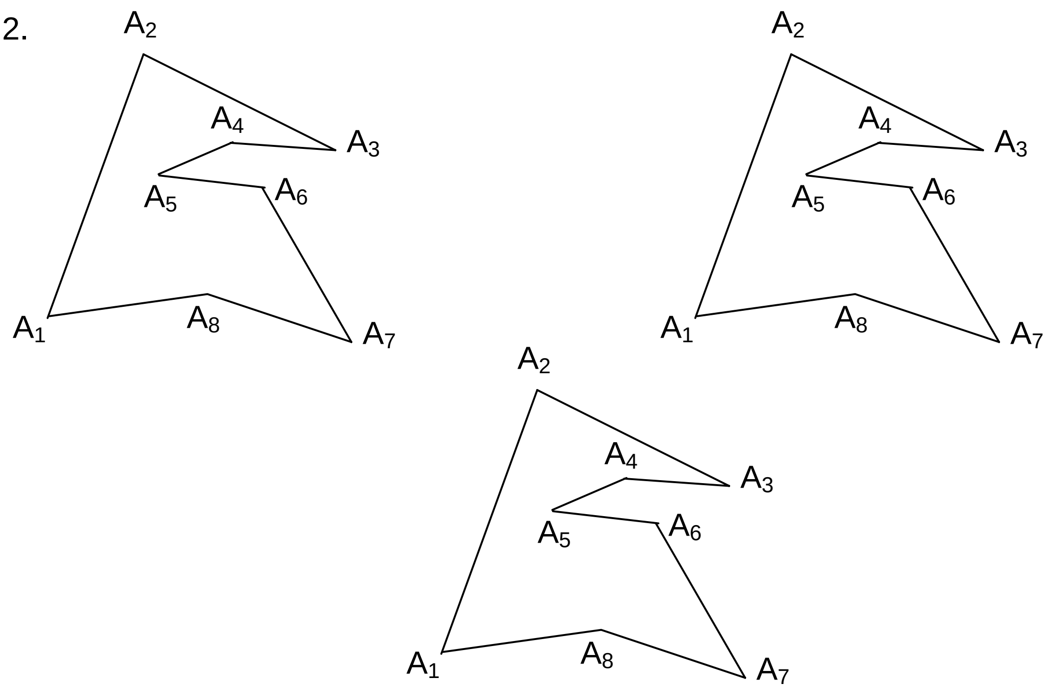
Tear off this page, make it the first page of your answers, and draw your answers for problems 1 through 3 on the figures below.

Problem 1.



\# of diagonals:

Problem 2.



Problem 3.