Pick's Theorem

Lattice points are points of the plane whose (x,y) coordinates are integers (Figure 1). For example, (3,2), (-2, -1), (-1,0), and (0, 0) are lattice points.

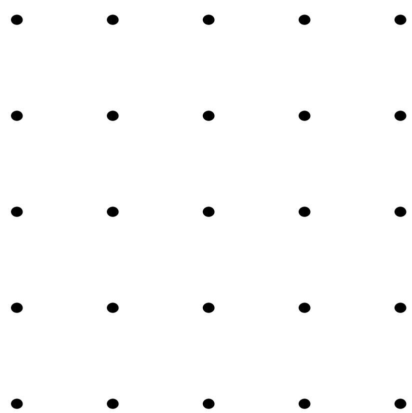

Figure 1

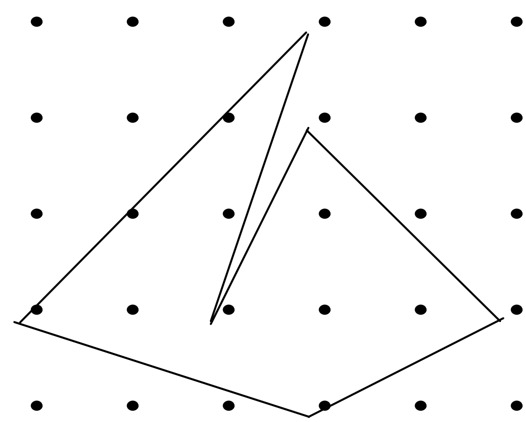

Figure 2

Let

$$A_1, A_2, \ldots, A_n \quad (1)$$

be lattice points for an integer $n \geq 3$. Traveling along the segments

$$A_1A_2, A_2A_3, \ldots, A_{n-1}A_n, A_nA_1 \quad (2)$$

in order gives a path from A_1 back to itself. Assume that this path does not intersect itself except for starting and finishing at A_1. Then we say that we have a lattice polygon with vertices (1) and sides (2).

In short, a lattice polygon is a polygon whose vertices are lattice points. We require the sides to form a single path that intersects itself only by starting and ending at the same point. Figure 2 shows a lattice polygon, but Figure 3 does not.

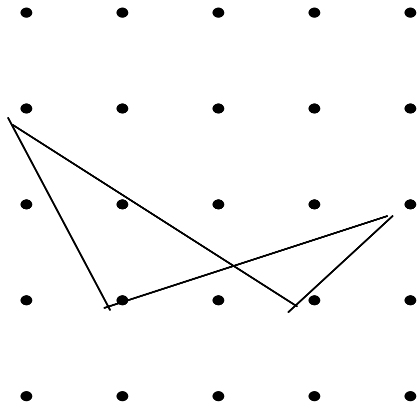

Figure 3

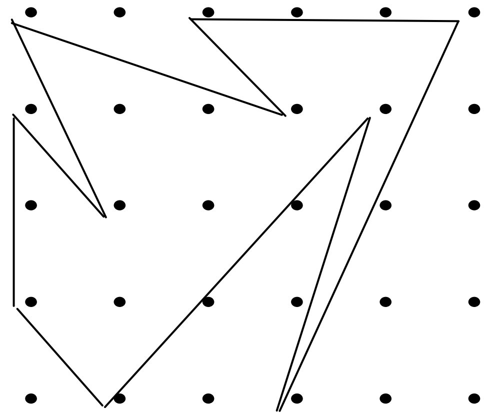

Figure 4

This essay proves the following result.

Pick's Theorem. A lattice polygon P has area $i + (s/2) - 1$, where i is the number of lattice points inside P and s is the number of lattice points on the sides of P.

For example, the lattice polygon in Figure 2 has 5 lattice points inside it and 9 lattice points on its sides (including the vertices), and so it has area $5 + (9/2) - 1 = 8½$.

Problem 1. For the lattice polygon P in Figure 4, find the number i of lattice points inside P and the number s of lattice points on the sides of P. Use Pick's Theorem to find the area of P.

For any lattice polygon P, let V(P) be the value of the quantity $i + (s/2) - 1$ in Pick's Theorem. For instance, we have V(P) = 8½ for the lattice polygon P in Figure 2. In words, V(P) is 1 less than the number of lattice points inside P plus half the number of lattice points on the sides of P.

A basic triangle T is a triangle that has lattice points as its three vertices and has no other lattice points inside it or on its sides (Figure 5).

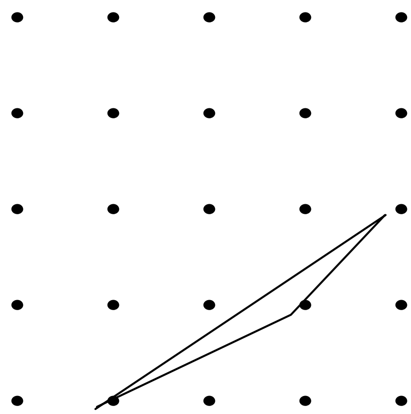

Figure 5

Problem 2. Why does V(T) = ½ for any basic triangle T?

Let P be a lattice polygon. Let T be a basic triangle that lies outside of P except for having one or two sides in common with P. Taking T and its interior and adding it to P and its interior gives a larger region whose perimeter is a lattice polygon Q. The sides of Q are the sides of P and T they do not have in common.

For example, the lattice polygon on the left of Figure 6 and the basic triangle on the right have one edge in common, and they combine to form the lattice polygon in Figure 7. The lattice polygon on the left of Figure 8 and the basic triangle on the right have two edges in common, and they combine to form the lattice polygon in Figure 9.

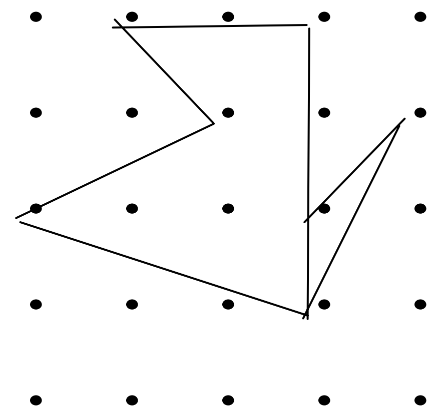

Figure 6

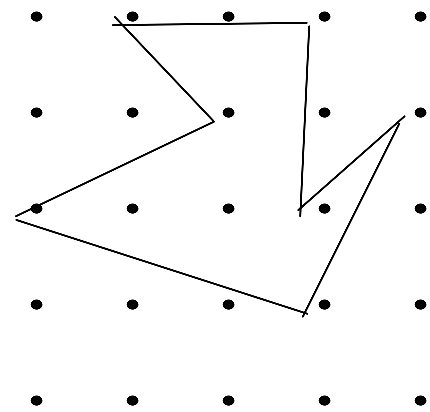

Figure 7

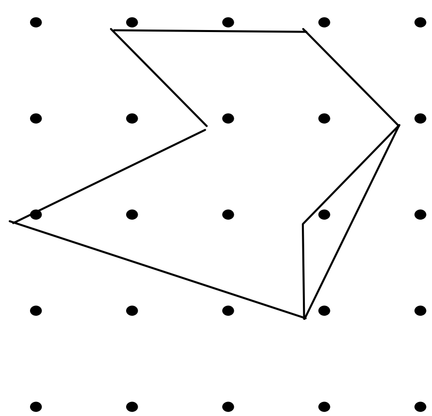

Figure 8

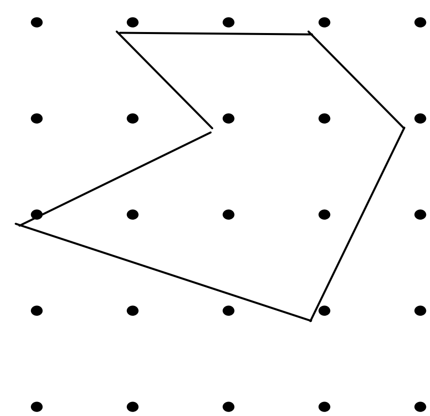

Figure 9

<u>Problem 3.</u> Let P, T, and Q be as in the first paragraph after Problem 2. Prove that V(Q) = V(P) + ½. Divide your answer into two cases, depending on whether P and T share one or two sides.

We use the following observation without proof.

<u>Claim 1.</u> Any lattice polygon can be divided into basic triangles whose interiors do not overlap.

Figure 10 illustrates Claim 1 by showing one way to divide up the lattice polygon in Figure 2.

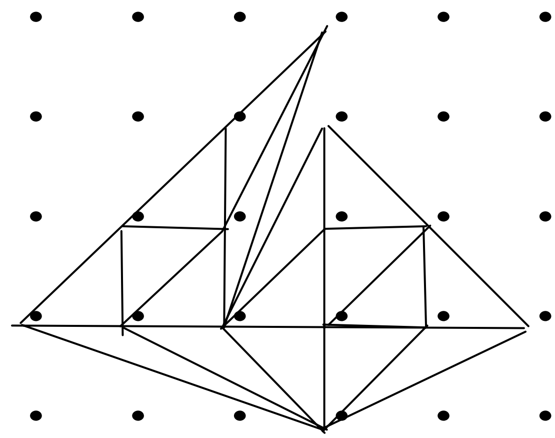

Figure 10

Problem 4. Show one way to divide the lattice polygon in Figure 4 into basic triangles whose interiors do not overlap. Page 9 contains a copy of Figure 4 to use. Tear off page 9 and turn it in as your answer to this problem.

Problem 5. Let P be a lattice polygon that is divided into *n* basic triangles whose interiors do not overlap. Deduce from Problems 2 and 3 that $V(P) = \frac{n}{2}$.

We are going to prove the following result.

Claim 2. Every basic triangle has area ½.

Problem 6. Deduce Pick's Theorem from Claims 1 and 2 and Problem 5.

A basic parallelogram is a parallelogram that has lattice points as its four vertices and has no other lattice points inside it or on its edges (Figure 11).

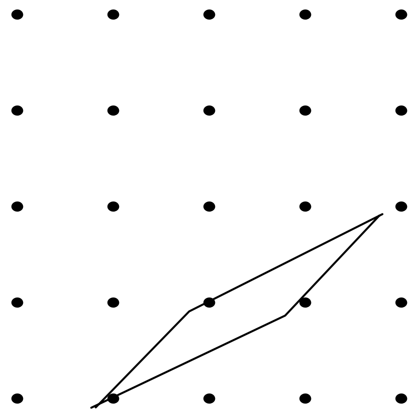

Figure 11

The following claim is the analogue of Claim 2 for parallelograms.

<u>Claim 3.</u> Every basic parallelogram has area 1.

The next problem shows that Claim 2 follows from Claim 3.

<u>Problem 7.</u> Let T be a basic triangle.

(a) Prove that there is a basic parallelogram that consists of T and another basic triangle congruent to T (Figure 12).

(b) Deduce Claim 2 from (a) and Claim 3.

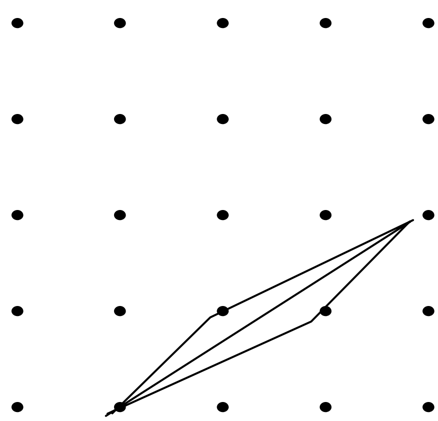

Figure 12

We end the Essay by proving Claim 3.

Let B be a basic parallelogram (Figure 13). We pick a vertex S of B and call it the base point of B. Translating S to another lattice point T translates B to another basic parallelogram C. We call T the base point of C.

Consider all the translations of B whose base points are lattice points (Figure 14). Like B, these translations are all basic parallelograms, and we include B among them. These basic parallelograms tile the plane: that is, they cover the plane and their interiors do not overlap. This follows from the fact that the four vertices of B are the only lattice points inside or on the sides of B.

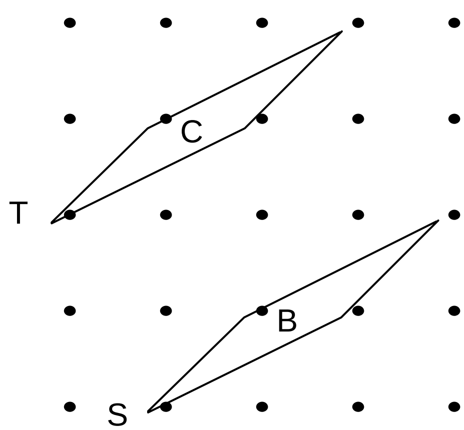

Figure 13

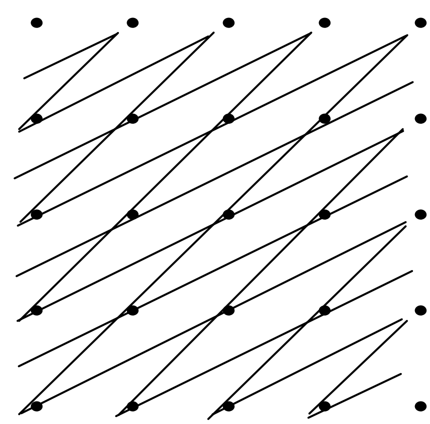

Figure 14

We want to find the area A of a basic parallelogram B. Let L be a positive integer greater than the distance between any two points of B. Take any integer n greater than $2L + 1$.

Problem 8. Consider all the translations of B whose base points are lattice points (j,k) for integers j and k from 1 through n.

(a) Why do these translates cover the entire square region enclosed by the lines

$x = 1 + L$, $x = n - L$, $y = 1 + L$, and $y = n - L$?

(b) Deduce that $n^2 A \geq (n - 2L - 1)^2$.

Problem 9. Consider the same translations of B as in Problem 8.

(a) Why do all these translations lie entirely within the square region enclosed by the lines $x = 1 - L,\ x = n + L,\ y = 1 - L,\ and\ y = n + L$?

(b) Deduce that $n^2 A \leq (n + 2L - 1)^2$.

Problem 10. Conclude from Problems 8 and 9 that

$$1 - \frac{4L + 2}{n} + \frac{4L^2 + 4L + 1}{n^2} \leq A \leq 1 + \frac{4L - 2}{n} + \frac{4L^2 - 4L + 1}{n^2}.$$

Problem 11. In Problem 10, hold L fixed and let n vary over all integers greater than $2L + 1$. Why does it follow that $A = 1$?

Problem 11 establishes Claim 3. This completes the proof of Pick's Theorem except for Claim 1, which we invite you to prove after Math Field Day. We hope that you will send us a proof of Claim 1 to post on the Math Field Day website along with our answers to the Essay problems.

Use this copy of Figure 4 to do Problem 4. Tear off this page and turn it in as your answer to Problem 4.

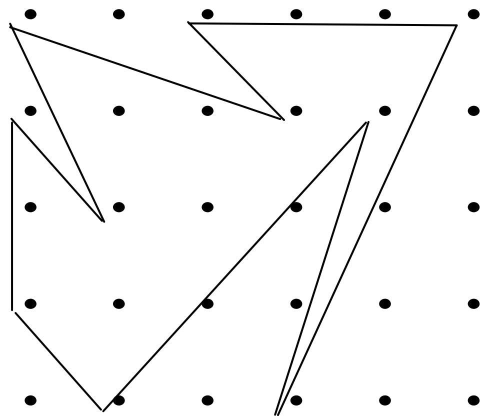

Figure 4