Fault-Free Tilings

We place all rectangles on the page so that they have horizontal and vertical sides. An m x n rectangle (read “m-by-n rectangle”) has height m and base n. We always take m and n to be positive integers.

A domino is a 1 x 2 or 2 x 1 rectangle. A domino is horizontal if it is 1 x 2 (Figure 1) and vertical if it is 2 x 1 (Figure 2).

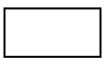

Figure 1

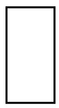

Figure 2

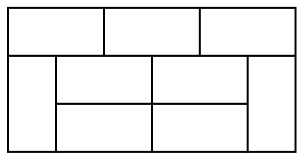

Figure 3

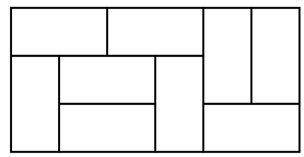

Figure 4

A tiling of a rectangle is a division of the rectangle into nonoverlapping dominoes. For example, Figure 3 shows one tiling of a 3 x 6 rectangle, and Figure 4 shows another.

A fault line of a tiled rectangle is a line that cuts the rectangle into two pieces without cutting any dominoes. For example, Figure 3 has a horizontal fault line 1 unit below the top edge. Figure 4 has a vertical fault line 4 units to the right of the left edge.

A tiling of a rectangle is fault-free if there is no fault line. For example, Figure 5 is a fault-free tiling of a 5 x 6 rectangle. Figure 6 is a fault-free tiling of a 6 x 8 rectangle.

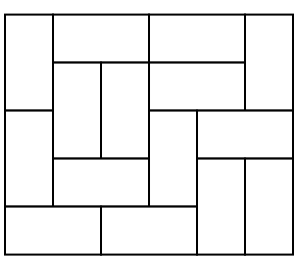

Figure 5

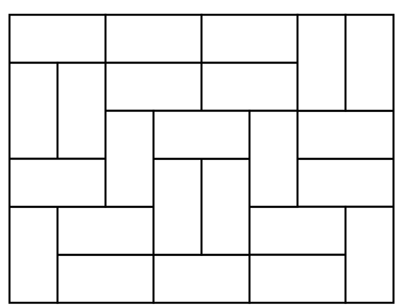

Figure 6

Which rectangles have fault-free tilings? Ron Graham discovered the answer and proved the following result. This essay outlines the proof.

Theorem. Let R be an m x n rectangle of area greater than 2. Then R has a fault-free tiling if and only if mn is even, $m \geq 5$, $n \geq 5$, and m and n do not both equal 6.

Problem 1. Consider a tiled rectangle of area greater than 2. If all the dominoes on the left edge of the rectangle face the same direction, prove that the tiling has a fault line. (Hint: Figures 7 – 10 suggest possibilities to consider. Figures 7 and 8 show 5 x 8 and 5 x 2 rectangles where all dominoes on the left edge are horizontal. Figures 9 and 10 show 6 x 3 and 6 x 1 rectangles where all dominoes on the left edge are vertical.)

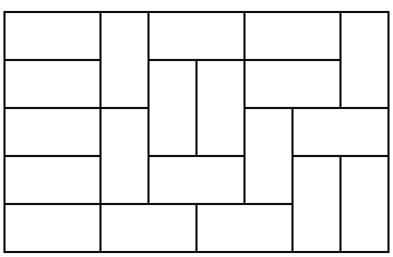
Figure 7

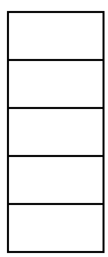
Figure 8

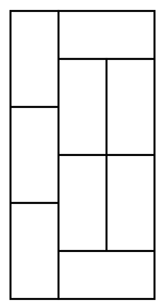
Figure 9

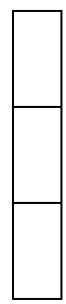
Figure 10

Consider a fault-free tiling of an m x n rectangle for $m \geq 3$. The left edge of the tiling includes a vertical domino D, by Problem 1 (Figure 11). Remove D (Figure 12), add horizontal rectangles all along the left boundary (Figure 13), and place a vertical domino E in the notch on the left boundary (Figure 14). This gives a fault-free tiling of an m x (n + 2) rectangle. In short, if an m x n rectangle has a fault-free tiling for $m \geq 3$, then so does an m x (n + 2) rectangle.

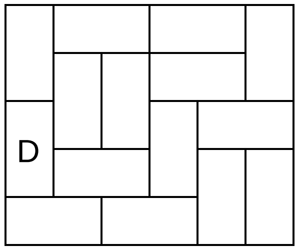

Figure 11

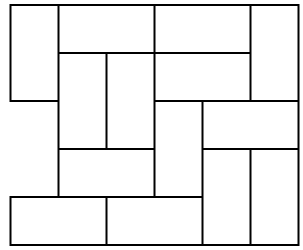

Figure 12

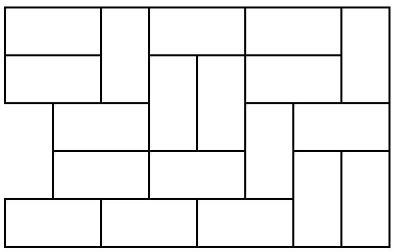

Figure 13

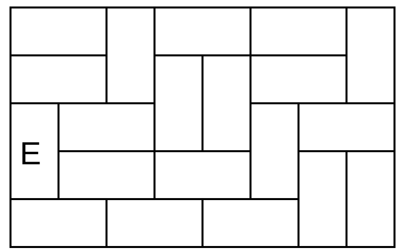

Figure 14

Problem 2. Draw a fault-free tiling of a 6 x 7 rectangle. Use the rectangle on p. 7, which is divided into unit squares by dotted lines. Tear off p. 7 and turn it in as your answer to this question. (Hint: Rotate Figure 5 by 90^0 and use the discussion of Figures 11-14.)

Problem 3. Let R be an m x n rectangle such that mn is even, m ≥ 5, n ≥ 5, and m and n do not both equal 6. Prove that R has a fault-free tiling. (Hint: Combine Figures 5 and 6 with 90^0 rotations and repeated applications of the discussion of Figures 11-14.)

We must show that the rectangles in Problem 3 are the only rectangles of area greater than 2 that have fault-free tilings.

Problem 4. If an m x n rectangle has a tiling, why must mn be even?

Consider a 1 x n or 2 x n rectangle of area greater than 2. By Problem 1, any tiling of the rectangle has a fault line, since all dominoes on the left edge of the tiling face the same way.

Suppose that Figure 15 is the left edge of a tiling of a 3 x n rectangle. If the tiling continues as in Figure 16, there is either a horizontal fault line 1 unit below the top edge or a vertical fault line 2 units to the right of the left edge, depending on whether the base of the rectangle is 2 or greater than 2. The only other possibility is for the tiling to continue as in Figure 17.

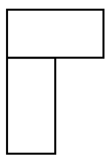

Figure 15

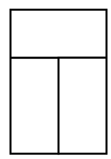

Figure 16

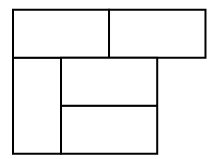

Figure 17

If the tiling in Figure 17 continues as in Figure 18, then the tiling has either a horizontal fault line 1 unit below the top edge or a vertical fault line 4 units to the right of the left edge, depending on whether the base of the rectangle is 4 or greater than 4. The only other possibility is for the tiling to continue as in Figure 19.

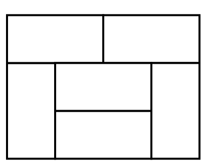

Figure 18

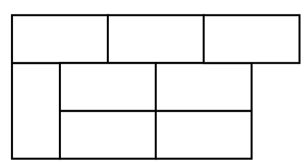

Figure 19

We continue to repeat the same reasoning as we follow the tiling to the right. Because the process cannot continue past the right edge of the tiling, we end by finding a fault line as in the discussions of Figures 16 and 18. This proves the following claim: if a tiling of a 3 x n rectangle has Figure 15 as its left edge, then the tiling has a fault line.

Reflecting Figure 15 across a horizontal line gives Figure 20. If we also reflect Figures 16 – 19 across horizontal lines, the accompanying discussion shows that, if a tiling of a 3 x n rectangle has Figure 20 as its left edge, then the tiling has a fault line.

Any tiling of a 3 x n rectangle has Figure 15, 20, or 21 as its left edge. If the tiling has Figure 21 as its left edge, the tiling has a fault line, by Problem 1. Together with the last two paragraphs, this shows that any tiling of a 3 x n rectangle has a fault line.

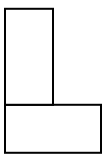

Figure 20

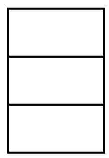

Figure 21

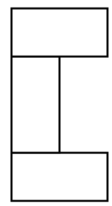

Figure 22

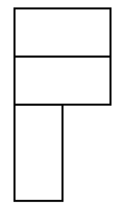

Figure 23

Problem 5. Suppose that a tiling of a 4 x n rectangle has Figure 22 as its left edge. Prove that the tiling has a fault line.(Hint: Adapt the discussion of Figures 15 – 19.)

Problem 6. Suppose that a tiling of a 4 x n rectangle has Figure 23 as its left edge. Prove that the tiling has a fault line.

(Hint: Adapt the discussion of Figures 15 – 19.)

Problem 7. Prove that any tiling of a 4 x n rectangle has a fault line.

(Hint: Combine Problems 1, 5, and 6 and reflection across a horizontal line.)

If an m x n rectangle has a fault-free tiling, the last two pages show that mn is even and m is at least 5. Rotating the rectangle 90^0 shows that n is also at least 5. It only remains for us to show that m and n cannot both be 6.

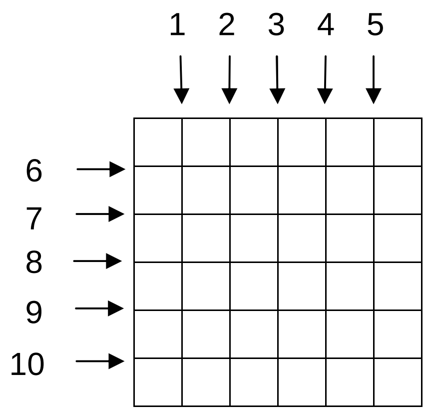

Figure 24

Figure 24 shows a 6 x 6 rectangle divided into squares of unit length. Any domino in a tiling of the rectangle is cut in half by exactly one of the lines numbered 1 – 10: each horizontal domino is cut in half by one of the vertical lines 1 – 5, and each vertical domino is cut in half by one of the horizontal lines 6 – 10.

Problem 8. Consider a tiling of the 6 x 6 rectangle in Figure 24. Let L be one of the lines numbered 1 – 10. Prove that L cuts in half an even number of dominoes in the tiling. (Hint: Consider the number of unit squares in Figure 24 on one side of L.)

Problem 9. Consider any tiling of the 6 x 6 rectangle in Figure 24. Prove that the tiling has a fault line. (Hint: By counting the total number of dominoes in the tiling, deduce that at least one of the lines 1 – 10 in Figure 24 cuts fewer than 2 dominoes. Then apply Problem 8.)

We have proved Graham’s theorem.

Draw a fault-free tiling of the 6 x 7 rectangle below. The dotted lines divide the rectangle into unit squares. Tear off this page and turn it in as your answer to Problem 2.