TRADING CHIPS

We start with a pile of poker chips. Each chip is red, white, or blue. A trade consists of taking two chips of different colors out of the pile and replacing them with two chips of the third color. For example, we can take a white and a blue chip out of the pile and replace them with two red ones.

Using only trades of this type, we want to make all the chips in the pile the same color. When is this possible? How do we proceed?

Suppose that there are different numbers of chips of each color. In particular, this means that at most one color is missing from the pile. The basic step consists of trading in one chip from each of the two most common colors for two chips of the least common color. For example, if we have 8 red, 5 white, and 6 blue chips, the basic step trades one red and one blue chip for two white chips to give 7 red, 7 white, and 5 blue chips.

The standard procedure consists of repeating the basic step until we reach one of the following two possibilities:

(A) two or more colors have the same number of chips, or

(B) the numbers of chips of the three colors are three consecutive integers.

We record the steps of the basic procedure as the rows of a table. The first three entries of each row are the numbers of red, white, and blue chips. The fourth entry contains the colors of the chips to be traded in.

R	W	B	Trade-in
0	4	7	WB
2	3	6	WB
4	2	5	RB
3	4	4	

Table 1

R	W	B	Trade-in
11	3	7	RB
10	5	6	RB
9	7	5	RW
8	6	7	

Table 2

For example, Table 1 shows the steps of the standard procedure if we start with 0 red, 4 white, and 7 blue chips, as in the first row. The two most common colors are white and blue, and we trade in 1 chip of each of these colors for two red chips. This gives 2 red, 3 white, and 6 blue chips, as in the second row. White and blue are still the two most common colors, and we trade in 1 white and 1 blue chip for two red ones. This gives 4 red, 2 white, and 5 blue chips, as in the third row. The two most common colors are now red and blue, and we trade in 1 red and 1 blue chip for two white ones. This gives 3 red, 4 white and 4 blue chips, as in the fourth row. The standard procedure ends here because there are equal numbers of white and blue chips.

Problem 1. Suppose we start with 3 red, 9 white, and 1 blue chip. Construct a table like Table 1 that shows the steps of the standard procedure and ends with possibility A.

Likewise, Table 2 shows the standard procedure when we start with 11 red, 3 white and 7 blue chips. The procedure ends when the numbers of chips of the three colors are the three consecutive integers 6, 7, and 8.

Problem 2. Suppose we start with 10 red, 5 white, and 0 blue chips. Construct a table like Table 2 that shows the steps of the standard procedure and ends with possibility B.

Note three properties of Table 1. First, the color that starts out as the most common remains the most common until the last row. Second, the number of chips of that color decreases by 1 each row. Third, the difference between the largest and smallest numbers in each row is at least 3 until the last row: those differences are 7 – 0 = 7 for row 1, 6 – 2 = 4 for row 2, and 5 – 2 = 3 for row 3. The next two problems depend on showing that these three properties always hold.

Problem 3. Assume that the numbers of chips of the three colors are all different and that the largest is at least 3 more than the smallest. Apply the basic step and suppose that the numbers of chips of the three colors are still all different. Show that the most common color was not changed by the basic step and that the number of chips of that color decreased by 1.

Problem 4. Deduce from Problem 3 that the standard procedure ends after a finite number of steps by reaching (A) or (B).

When we get the same number of chips of two colors, as in (A), we can trade in pairs of those chips to make all the chips in the pile the same color. For example, because Table 1 ended with 4 white and 4 blue chips, we can trade in 4 pairs of white and blue chips to make all the chips in the pile red. Table 3 completes Table 1 to show how, starting with 0 red, 4 white, and 7 blue chips, we can perform trades to make the pile all red.

R	W	B	Trade-in
0	4	7	WB
2	3	6	WB
4	2	5	RB
3	4	4	WB
5	3	3	WB
7	2	2	WB
9	1	1	WB
11	0	0	

Table 3

Problem 5. As we extended Table 1 to Table 3, copy your answer to Problem 1 and extend it to a table that shows all the trades taking a pile of 3 red, 9 white, and 1 blue chip to a pile of one color.

The key to the next problem is showing that the sum of three consecutive integers is always a multiple of 3.

Problem 6. If the total number of chips in the pile is not a multiple of 3, deduce from Problem 4 that it is always possible to make trades so that all chips in the pile are one color, regardless of the colors of the original chips in the pile.

Possibility (B) can arise when the number of chips in the pile is a multiple of 3. When (B) arises, the next two problems show that no sequence of trades can make the original pile of chips one color. Note that we allow all trades described in the first paragraph of the Essay, not just basic steps.

Suppose we have r red, w white, and b blue chips. We consider the differences

$$r - w,\ w - b,\ b - r. \qquad (1)$$

For example, row 1 of Table 2 gives differences

$$\begin{aligned} r - w &= 11 - 3 = 8, \\ w - b &= 3 - 7 = -4, \\ b - r &= 7 - 11 = -4. \end{aligned} \qquad (2)$$

The second row of Table 2 gives differences

$$\begin{aligned} r - w &= 10 - 5 = 5, \\ w - b &= 5 - 6 = -1, \\ b - r &= 6 - 10 = -4. \end{aligned} \qquad (3)$$

Subtract each difference in (2) from the corresponding difference in (3). This shows that going from row 1 to row 2 changes the difference $r - w$ by $5 - 8 = -3$, $w - b$ by $-1 - (-4) = 3$, and $b - r$ by $-4 - (-4) = 0$. The next problem shows that these changes are typical.

Problem 7. Show that any trade changes each difference in (1) by 3, 0, or –3.

Applying Problem 7 repeatedly shows that every sequence of trades changes the differences in (1) by multiples of 3. This observation is the key to the next problem.

Problem 8. Suppose that applying the standard procedure to a pile of chips produces possibility B. Deduce from the observation after Problem 7 that no sequence of trades could make the original pile of chips one color.

In short, the standard procedure settles the question of making a pile of chips one color. The procedure always ends with (A) or (B), by Problem 4. If it ends with (A), the pile of chips can be made one color, as in Problem 5. In possibility (B), there is no sequence of trades that makes the pile one color, by Problem 8.

When the number of chips in the pile is not a multiple of 3, the pile can be made one color, by Problem 6. The differences in (1) show immediately what this color is.

For example, suppose we start with 13 red, 2 white, and 8 blue chips. The total number of chips is $13 + 2 + 8 = 23$, which is not a multiple of 3. Computing the differences gives

$$r - w = 13 - 2 = 11,$$

$$w - b = 2 - 8 = -6,$$

$$b - r = 8 - 13 = -5.$$

Only one of these differences, $w - b = -6$, is a multiple of 3. Because this difference does not involve the color red, the next problem shows that red is the one color that every chip can be made.

Problem 9. Suppose that the number of chips in the pile is not a multiple of 3. Show that exactly one of the differences in (1) is a multiple of 3. Show that the color not involved in this difference is the one color that all the chips can be made.

(Hint: Use Problem 6 and the observation after Problem 7.)