Rectangular Tilings

An m × n rectangle (read "m–by–n rectangle") has vertical sides of length m and horizontal sides of length n. We always take m and n to be positive integers.

A rectangle has a (c,d) tiling if we can divide it into nonoverlapping c × d and d × c subrectangles. For example, Figure 1 shows a (3,1) tiling of a 9 × 7 rectangle, and Figure 2 shows a (3,2) tiling of an 11 × 18 rectangle.

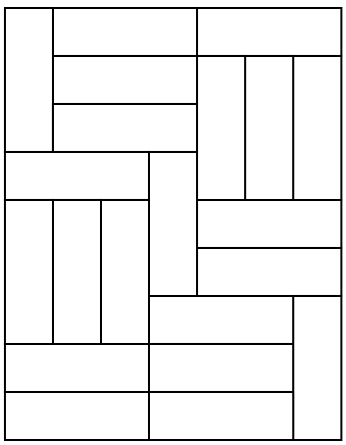


Figure 1

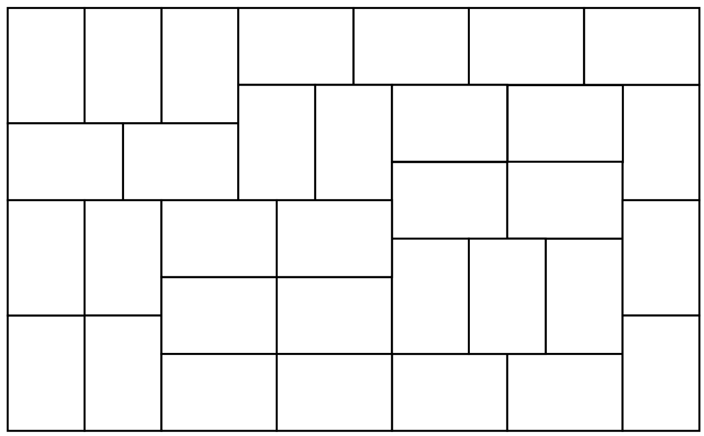


Figure 2

For a (c,d) tiling of a rectangle, we say that the c × d subrectangles are tiles of orientation c × d and that the d × c subrectangles are tiles of orientation d × c. For example, in Figure 1, the vertical tiles have orientation 3 × 1, and the horizontal tiles have orientation 1 × 3.

We want to determine which rectangles have (c,d) tilings. The unit squares of a rectangle are the squares of a (1,1) tiling. For any (c,d) tiling of a rectangle, the tiles are made up of unit squares.

The first two problems ask you to find two tilings, which turn out to be much simpler than those in Figures 1 and 2. Use the rectangles on p. 7 to draw the tilings. Tear off p. 7 and turn it in to answer Problems 1 and 2.

Problem 1. Draw a (2,5) tiling of the 6 × 10 rectangle on the top of p. 7.

Problem 2. Draw a (2,5) tiling of the 9 × 10 rectangle on the bottom of p. 7.

Here is an observation about the simplest kind of tiling: If m is a multiple of c and n is a multiple of d, then an m × n rectangle has a (c,d) tiling where all the tiles have orientation c × d. In fact, if m = cs and n = dt for integers s and t, stacking s tiles in orientation c × d in a column fills out the height m = cs of the rectangle, and placing t of these columns side by side fills out the length n = dt of the rectangle. Figure 3 illustrates this for m = 9, c = 3, n = 7, and d = 1.

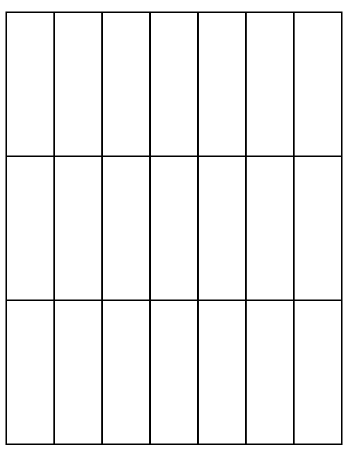


Figure 3

The second simplest kind of tiling is one where the tiles of each orientation form a rectangle. For example, Figure 4 shows a (3,2) tiling of an 11×18 rectangle where the tiles of orientation 3×2 form a 3×18 rectangle and the tiles of orientation 2×3 form an 8×18 rectangle. Note that the length 18 of the rectangle is a multiple of both 3 and 2 and that the height 11 of the rectangle satisfies 11 = 3(1) + 2(4), where 1 and 4 are the numbers of rows of tiles with each orientation.

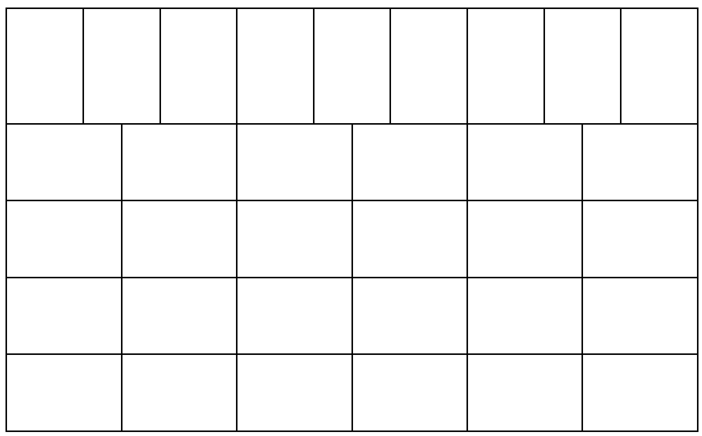


Figure 4

Problem 3 generalizes this example and gives conditions that allow the second simplest kind of tiling.

Problem 3. If n is multiple of both c and d, and if m equals cs + dt for positive integers s and t, show that an $m \times n$ rectangle has a (c,d) tiling where the tiles of each orientation form a rectangle.

This essay provides a proof of the following result discovered in 1969 by D. A. Klarner.

Rectangular Tiling Theorem. An m × n rectangle has a (c,d) tiling if and only if condition (i) or condition (ii) below holds.

(i) Either m or n is a multiple of c, and the other is a multiple of d.

(ii) Either m or n is a multiple of both c and d, and the other equals cs + dt for positive integers s and t.

Together with Problem 3 and the observation after Problem 2, the Rectangular Tiling Theorem has a striking consequence: if a rectangle has a (c,d) tiling, then it has a (c,d) tiling where either all the tiles have the same orientation or the tiles of each orientation form a rectangle. For example, a 9 × 7 rectangle, which has the (3,1) tiling in Figure 1, also has the (3,1) tiling in Figure 3. Likewise, an 11 × 18 rectangle, which has the (3,2) tiling in Figure 2, also has the (3,2) tiling in Figure 4.

We proceed towards a proof of the Rectangular Tiling Theorem.

Problem 4. If an m × n rectangle has a (c,d) tiling, why does m = cs + dt for nonnegative integers s and t? (Hint: Consider the tiles along a vertical side of the rectangle.)

In Problems 5–8, consider a (c,d) tiling of an m × n rectangle. Use c colors to shade the vertical columns of unit squares in the rectangle: use each color for one of the first c columns, use the colors in the same order for the next c columns, and continue to repeat the colors in the same order until all columns are colored. For example, if we start with the (c,d) = (3,1) tiling of the 9 × 7 rectangle in Figure 1, we use c = 3 colors, repeated cyclically, for the columns of unit squares in the rectangle, as in Figure 5, where R, Y, and G represent red, yellow, and green.

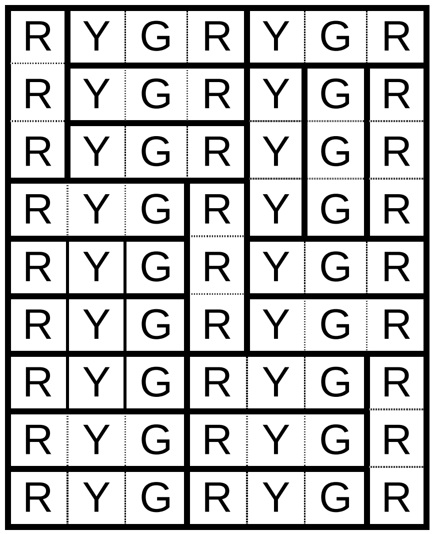


Figure 5

Given a (c,d) tiling of an $m \times n$ rectangle, we have now colored the unit squares. Let *x* be the number of unit squares in the rectangle that have the color used in the first column. Let *y* be the number of unit squares in the rectangle that have the color used in column c (where y = 0 if n < c).

Problem 5. If n is not a multiple of c, why does x – y = m?

Problem 6. (a) Consider the number of unit squares of any one color in a tile of orientation c × d. Why is this number a multiple of c?

(b) Why does a tile of orientation d × c contain exactly d unit squares of each color?

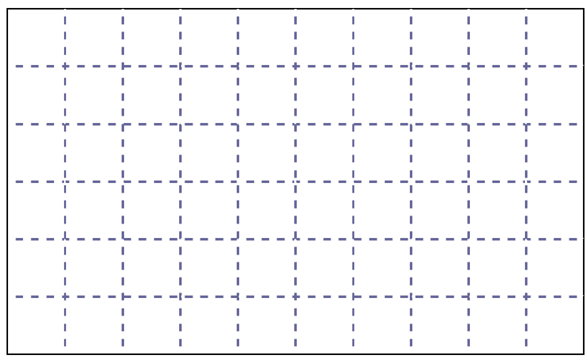
Problem 7. Let the tiling have w tiles of orientation $d \times c$. Deduce from Problem 6 that $x = cu + dw$ and $y = cv + dw$ for integers u and v.

Problem 8. Deduce from Problems 5 and 7 that m or n is a multiple of c.

Problem 9. Use Problems 3, 4, and 8 and the observation after Problem 2 to prove the Rectangular Tiling Theorem.

Tear off this page and turn it in to answer Problems 1 and 2.

Problem 1. Draw a (2,5) tiling of the 6 × 10 rectangle below. The rectangle is divided into unit squares by dotted lines.



Problem 2. Draw a (2,5) tiling of the 9 × 10 rectangle below. The rectangle is divided into unit squares by dotted lines.