

Fibonacci Numbers and Greatest Common Divisors

The Fibonacci numbers are the numbers in the sequence

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \dots$$

After starting with two 1s, we get each Fibonacci number by adding the two previous ones. Thus, after the two initial 1s, we get $1 + 1 = 2$, $1 + 2 = 3$, $2 + 3 = 5$, $3 + 5 = 8$, and so on.

Let f_n be the n^{th} Fibonacci number. We have found that

$$f_1 = 1, f_2 = 1, f_3 = 2, f_4 = 3, f_5 = 5, f_6 = 8, f_7 = 13, f_8 = 21, f_9 = 34, f_{10} = 55, f_{11} = 89, \\ f_{12} = 144.$$

Problem 1. Find the next four Fibonacci numbers f_{13} , f_{14} , f_{15} , and f_{16} .

Each Fibonacci number from the third one on is the sum of the two preceding ones. This is the relation

$$f_{n+2} = f_n + f_{n+1} \quad (1)$$

for all integers $n \geq 1$.

The Fibonacci numbers are also related to each other in surprising ways. Two relations important to us are

$$f_{b+c+1} = f_b f_c + f_{b+1} f_{c+1} \quad (2)$$

and

$$f_{b+1} f_{c+1} - f_b f_{b+c+1} = (-1)^b f_c \quad (3)$$

for all positive integers b and c . For example, setting $b = 3$ and $c = 7$ in the right-hand side of (2) gives

$$f_3 f_7 + f_4 f_8 = 2(13) + 3(21) = 26 + 63 = 89.$$

Since $89 = f_{11} = f_{3+7+1}$, (2) checks for $b = 3$ and $c = 7$.

Problem 2. Check (3) for $b = 9$ and $c = 6$.

The last five problems in the essay show that (2) and (3) hold for all positive integers b and c .

The greatest common divisor of positive integers a and b is the largest positive integer d such that a and b are both multiples of d . Let (a,b) represent the greatest common divisor of a and b . For example, we have $(24,42) = 6$ because 24 and 42 are both multiples of 6 and no larger integer. Another way to say that 24 and 42 are both multiples of 6 is to say that they are both evenly divisible by 6, which explains the term “greatest common divisor.” We have $(24,35) = 1$ because 24 and 35 are not both multiples of any integer larger than 1. Note also that $(24,24) = 24$.

This essay shows that

$$(f_a, f_b) = f_{(a,b)} \quad (4)$$

for all positive integers a and b . Thus, the greatest common divisor of any two Fibonacci numbers is a Fibonacci number, and its position in the sequence of Fibonacci numbers is the greatest common divisor of the positions of the first two. For example, when $a = 8$ and $b = 12$, Equation 4 says that

$$(f_8, f_{12}) = (21, 144) = 3 = f_4 = f_{(8,12)}.$$

Setting $a = 24$ and $b = 42$ in (4) shows that

$$(f_{24}, f_{42}) = f_{(24, 42)} = f_6 = 8.$$

This is certainly an easier way to find the greatest common divisor of f_{24} and f_{42} than by computing that the 24th Fibonacci number is 45,368, the 42nd Fibonacci number is 267,914,296 is 8.

Problem 3. Use (4) to find the greatest common divisor of f_{63} and f_{98} .

A key fact in working with greatest common divisors is that

$$(a,b) = (a-b,b) \quad (5)$$

for all positive integers a and b with $a > b$. This says that the greatest common divisor of two numbers stays the same when we subtract the smaller number from the larger. For example, this lets us quickly find the greatest common divisor of 4017 and 4011:

$$(4017, 4011) = (4017-4011, 4011) = (6,4011) = 3.$$

Equation 5 follows from two observations. First, if a and b are both multiples of a positive integer d , then so is $a - b$: if $a = ds$ and $b = dt$ for integers s and t , it follows that $a - b = ds - dt = d(s-t)$. Second, if $a - b$ and b are both multiples of a positive integer d , then so is a : if $a - b = du$ and $b = dt$ for positive integers u and t , it follows that

$$a = (a - b) + b = du + dt = d(u + t).$$

The next two problems show that

$$(f_a, f_b) = (f_{a-b}, f_b) \quad (6)$$

for all positive integers a and b with $a > b$. This is the analogue of (5) for Fibonacci numbers.

Problem 4. Let a and b be positive integers with $a > b$. If f_a and f_b are both multiples of a positive integer d , prove that f_{a-b} is as well. Specifically, if $f_a = ds$ and $f_b = dt$ for integers s and t , show that f_{a-b} is d times an integer by substituting $a - b$ for c in (3).

Problem 5. Let a and b be positive integers with $a > b$. If f_{a-b} and f_b are both multiples of a positive integer d , prove that f_a is as well. Specifically, when $a \geq b + 2$, if $f_{a-b} = du$ and $f_b = dt$ for integers u and t , show that f_a is d times an integer by substituting $a - b - 1$ for c in (2). When $a = b + 1$, argue directly.

Problems 4 and 5 show that (6) holds for all positive integers a and b . Combining (5) and (6) lets us prove that (4) holds for all positive integers a and b .

Problem 6. Let a and b be positive integers. Suppose we know that

$$(f_c, f_d) = f_{(c,d)}$$

for all positive integers c and d such that $c + d < a + b$. Deduce that

$$(f_a, f_b) = f_{(a,b)} \tag{7}$$

by using Equations 5 and 6 if $a > b$, interchanging a with b if $a < b$, and arguing directly if $a = b$.

Using Problem 6 to work upwards through all positive integral values of $a + b$ gives Equation 4 for all positive integers a and b , which was our objective.

It only remains for us to prove that (2) and (3) hold for all positive integers b and c . We start by showing that, when (2) holds for two consecutive integral values for c , then it also holds when c is the next largest integer.

Problem 7. Let k be a positive integer. If Equation 2 holds for $c = k$ and $c = k + 1$, deduce that it also holds for $c = k + 2$. In other words, show that the equations

$$f_{b+k+1} = f_b f_k + f_{b+1} f_{k+1}$$

and

$$f_{b+k+2} = f_b f_{k+1} + f_{b+1} f_{k+2}$$

imply that

$$f_{b+k+3} = f_b f_{k+2} + f_{b+1} f_{k+3}.$$

Use (1) for three values of n .

Problem 8. Show that Equation 2 holds for $c = 1$ and $c = 2$ by using Equation 1 as needed.

Problems 7 and 8 imply that Equation 2 holds for all positive integers b and c . Since (2) holds for $c = 1$ and $c = 2$ (by Problem 8), it also holds for $c = 3$ (by Problem 7). Since (2) holds for $c = 2$ and $c = 3$, it also holds for $c = 4$ (by problem 7). Continuing in this way shows that (2) holds for all positive integers c .

Equation 3 follows from Equation 2 and the fact that

$$f_{b+1} f_{b-1} - f_b^2 = (-1)^b \tag{8}$$

for every integer $b \geq 2$.

Problem 9. Let k be an integer with $k \geq 2$. Prove that the left-hand side of (8) for $b = k + 1$ is -1 times the left-hand side of (8) for $b = k$. In other words, prove that $f_{k+2} f_k - f_{k+1}^2$ is -1 times $f_{k+1} f_{k-1} - f_k^2$. Use (1) as necessary.

Problem 10. Check directly that (8) holds for $b = 2$. Combine this with Problem 9 to conclude that (8) holds for every integer $b \geq 2$.

Replacing b by $b - 1$ in (2) gives the relation

$$f_{b+c} = f_{b-1}f_c + f_b f_{c+1} \quad (9)$$

for all positive integers b and c with $b \geq 2$ (which makes $b - 1$ positive).

Problem 11. Derive Equation 3 as follows. When $b \geq 2$, use (9) and (2) to eliminate f_{b+c} and f_{b+c+1} from the left side of (3), and use (8) to simplify the result. When $b = 1$, check (3) directly.