<u>Rooks and Pascal's Triangle</u>

An (n,k) board consists of n^2 squares arranged in n rows and n columns with shading in the first k squares of the diagonal running downwards to the right. We consider (n,k) boards for all positive integers n and all integers k from 0 through n. For example, Figures 1a-d show (3,0), (3,1), (3,2), and (3,3) boards.

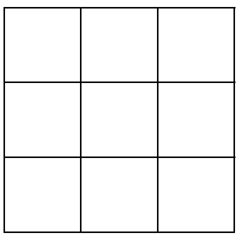


Figure 1a

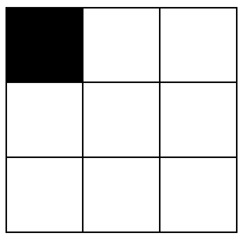


Figure 1b

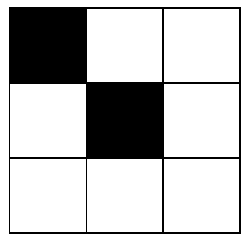


Figure 1c

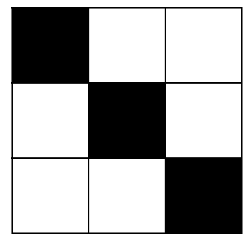


Figure 1d

"Placing rooks" on an (n,k) board means setting objects called "rooks" on unshaded squares of the board so that each row and column has exactly one rook. For instance, Figure 2 shows all the ways to place rooks on the (3,1) board in Figure 1b, where Xs mark the positions of the rooks.

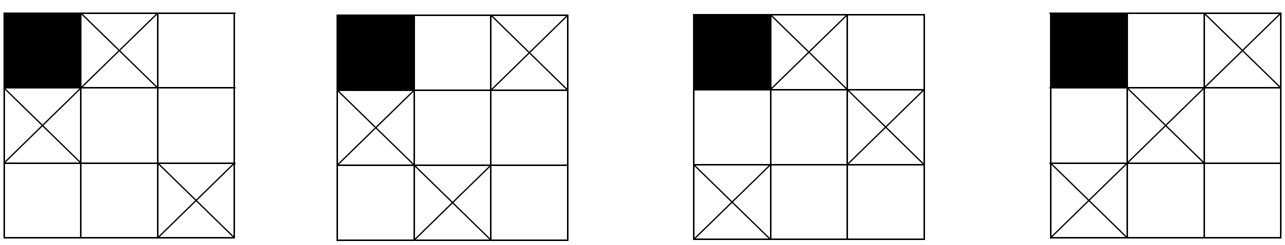


Figure 2

Let R(n,k) be the number of ways to place rooks on an (n,k) board. Figure 2 shows that R(3,1) = 4.

Problem 1. Demonstrate that R(3,0) = 6, R(3,2) = 3, and R(3,3) = 2 by showing all the ways to place rooks on the (3,0), (3,2), and (3,3) boards in Figures 1a, 1c, and 1d.

We call the objects we place "rooks" because rooks are chess pieces that attack along rows and columns of the chessboard. We think of setting rooks in different rows and columns as positioning them so they cannot attack each other.

We want an efficient way to find the numbers R(n,k). We proceed by working upwards through the possible values 0, 1, 2, …, n of the number k of shaded squares.

When k = 0, we place the rooks on a board without shaded squares. The rook in column 1 can go in any of the n rows, the rook in column 2 can go in any of the remaining n – 1 rows, the rook in column 3 can go in any of the remaining n – 2 rows, and so on until the rook in the last column goes into the 1 remaining row. Thus, there are $n(n-1)\cdots 1$ ways to place rooks on an (n,0) board, and we have

$$R(n,0) = n(n-1)\cdots 1 \qquad (1)$$

for $n \geq 1$. For instance, taking n = 3 in (1) gives $R(3,0) = 3\cdot 2\cdot 1 = 6$, as in Problem 1.

For any positive integer n, n! (read "n-factorial") denotes the product $n(n-1)\cdots 1$ of the integers from n down through 1. We think of the (0,0) board as having no squares, and we set R(0,0) = 1 and 0! = 1 because there is one way to do nothing. Combining the last sentence with Equation 1 shows that

$$R(n,0) = n! \qquad (2)$$

for all integers $n \geq 0$. For instance, setting n = 5 in Equation 2 shows that there are $5! = 5\cdot 4\cdot 3\cdot 2\cdot 1 = 120$ ways to place rooks on the (5,0) board in Figure 3.

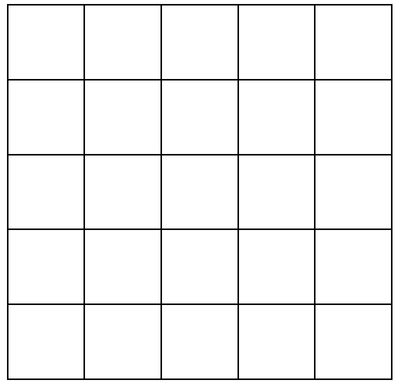


Figure 3

For $n \geq 1$, we divide the R(n,0) ways to place rooks on an (n,0) board into two groups, depending on whether or not there is a rook in row 1 of column 1. If there is, position other rooks so that exactly one of them lies in each of the remaining $n - 1$ rows and $n - 1$ columns. This is equivalent to placing rooks on an $(n - 1, 0)$ board, which can be done in R(n-1,0) ways. On the other hand, if there is no rook in row 1 of column 1, position rooks so that exactly one of them lies in each of the n rows and n columns and the first square of the diagonal is empty. This corresponds to placing rooks on an (n,1) board, which can be done in R(n,1) ways. Combining the two groups of rook placements shows that

$$R(n,0) = R(n-1,0) + R(n,1).$$

Rewriting this as $R(n,1) = R(n,0) - R(n-1,0)$ and substituting from Equation 2 shows that

$$R(n,1) = n! - (n-1)! \qquad (3)$$

for $n \geq 1$. For example, when $n = 3$, we get $R(3,1) = 3! - 2! = 6 - 2 = 4$, as in Figure 2.

Problem 2. If $n \geq 2$, explain why $R(n,1) = R(n-1,1) + R(n,2)$.

Then use Equation 3 to conclude that

$$R(n,2) = n! - 2(n-1)! + (n-2)! \tag{4}$$

For example, when $n = 3$, Equation 4 shows that

$$R(3,2) = 3! - 2(2!) + 1! = 6 - 4 + 1 = 3,$$

as in Problem 1.

Problem 3. If $n \geq 3$, explain why $R(n,2) = R(n-1,2) + R(n,3)$.

Then use Equation 4 to conclude that

$$R(n,3) = n! - 3(n-1)! + 3(n-2)! - (n-3)!. \tag{5}$$

For example, when $n = 3$, Equation 5 shows that

$$R(3,3) = 3! - 3(2!) + 3(1!) - 0! = 6 - 6 + 3 - 1 = 2,$$

as in Problem 1.

We collect Equations 2-5 below.

$$\begin{aligned}
R(n,0) &= n! \\
R(n,1) &= n! - (n-1)! \\
R(n,2) &= n! - 2(n-1)! + (n-2)! \\
R(n,3) &= n! - 3(n-1)! + 3(n-2)! - (n-3)!
\end{aligned} \tag{6}$$

The coefficients of the factorials in (6) seem to come from Pascal's triangle, whose top rows are as follows.

1		Row 0
1 1		Row 1
1 2 1		Row 2
1 3 3 1		Row 3
1 4 6 4 1		Row 4
1 5 10 10 5 1		Row 5

Each row of Pascal's triangle starts and ends with 1, and each other entry is the sum of the two entries above it. For example, Row 5 consists of the sums of adjacent entries in Row 4 -- namely, 1 + 4 = 5, 4 + 6 = 10, 6 + 4 = 10, and 4 + 1 = 5 -- with a 1 added on each end of the row.

Problem 4. Find Rows 6 and 7 of Pascal's triangle.

Let $\binom{k}{0}, \binom{k}{1}, \ldots, \binom{k}{k}$ denote the entries in row k of Pascal's triangle. For example, the entries in row 3 are

$$\binom{3}{0} = 1, \quad \binom{3}{1} = 3, \quad \binom{3}{2} = 3, \quad \binom{3}{3} = 1. \tag{7}$$

Comparing the equations in (6) with rows 0-3 of Pascal's triangle suggests that

$$R(n,k) = \binom{k}{0} n! - \binom{k}{1} (n-1)! + \binom{k}{2} (n-2)! - \ldots + (-1)^k \binom{k}{k} (n-k)! \tag{8}$$

for $0 \le k \le n$. For instance, taking k = 3 in Equation 8 gives

$$R(n,3) = \binom{3}{0} n! - \binom{3}{1} (n-1)! + \binom{3}{2} (n-2)! - \binom{3}{3} (n-3)!,$$

and substituting the values from (7) gives (5).

Taking k = 4 in (8) and substituting the values from Row 4 of Pascal's triangle gives

$$R(n,4) = n! - 4(n-1)! + 6(n-2)! - 4(n-3)! + (n-4)!. \quad (9)$$

Setting n = 4 shows that there are

$$R(4,4) = 4! - 4(3!) + 6(2!) - 4(1!) + 0!$$

$$= 24 - 4(6) + 6(2) - 4(1) + 1 = 9$$

ways to place rooks in the (4,4) square in Figure 4. Setting n = 6 in (9) shows that there are

$$R(6,4) = 6! - 4(5!) + 6(4!) - 4(3!) + 2!$$

$$= 720 - 4(120) + 6(24) - 4(6) + 2$$

$$= 720 - 480 + 144 - 24 + 2 = 362$$

ways to place rooks on the (6,4) square in Figure 5.

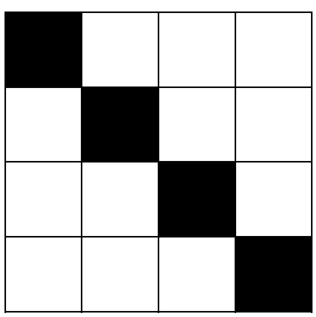


Figure 4

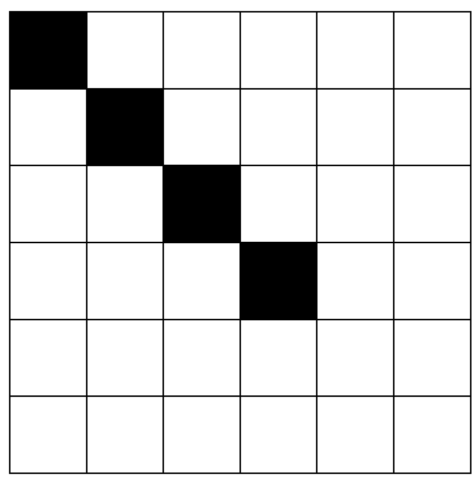
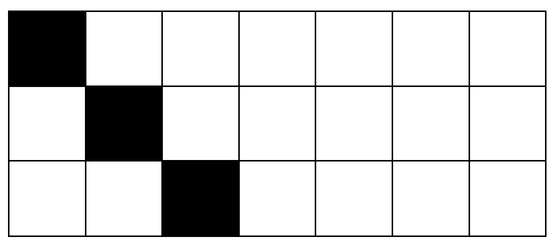


Figure 5

Problem 5. Use Equation 8 and Pascal's triangle to find the numbers R(5,5) and R(7,5) of ways to place rooks on the (5,5) and (7,5) boards in Figures 6 and 7.

Problem 6. Use Equation 8 and Problem 4 to find the number R(6,6) of ways to place rooks on the (6,6) square in Figure 8.

Problem 7. A teacher has one paper from each student in a class of six. The teacher hands one paper at random to each of the six students. Explain why $\frac{R(6,6)}{6!}$ is the probability that each student receives someone else's paper. Use Problem 6 to express this probability as a fraction in lowest terms.



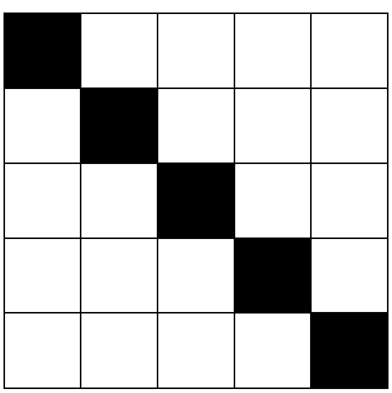


Figure 6

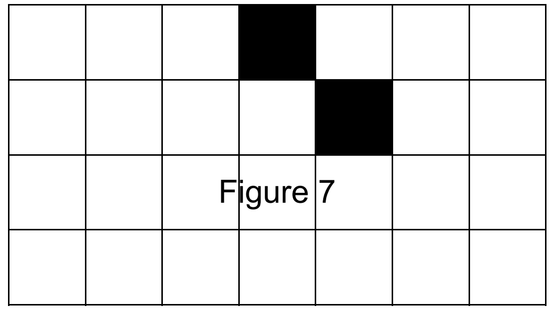


Figure 7

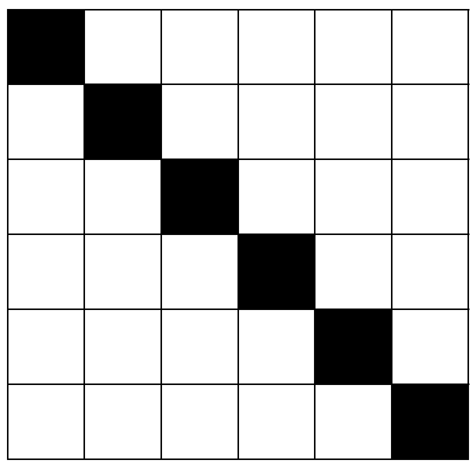


Figure 8

We must still prove that Equation 8 holds for all integers k and n with $0 \le k \le n$.

The next problem generalizes the first sentences of Problems 2 and 3.

Problem 8. Explain why R(n,k-1) = R(n-1,k-1) + R(n,k) for $1 \le k \le n$.

Recall that $\binom{k}{0}, \binom{k}{1}, ..., \binom{k}{k}$ are the entries in row k of Pascal's triangle. Because the row starts and ends with 1, we have

$$\binom{k}{0} = 1 = \binom{k}{k}. \qquad (10)$$

Because each other entry in the row is the sum of the two entries above it, we have

$$\binom{k}{s} = \binom{k-1}{s-1} + \binom{k-1}{s} \qquad (11)$$

for $1 \le s \le k-1$. For example, taking k = 5 and s = 2 gives $\binom{5}{2} = \binom{4}{1} + \binom{4}{2}$, which says that the first 10 in row 5 of Pascal's triangle is the sum of the 4 and 6 above it.

The next problem shows that we can increase the value of k in Equation 8 from each nonnegative integer to the next, since (13) is the right-hand side of Equation 8, and we get (12) from (13) by replacing k with k - 1.

Problem 9. Take an integer $k \ge 1$. Assume we know that R(n,k-1) equals

$$\binom{k-1}{0} n! - \binom{k-1}{1} (n-1)! + \binom{k-1}{2} (n-2)! - \ldots + (-1)^{k-1} \binom{k-1}{k-1} (n-k+1)! \qquad (12)$$

for every integer n such that $n \ge k - 1$. Use Problem 8 and Equations 10 and 11 to conclude that R(n,k) equals

$$\binom{k}{0} n! - \binom{k}{1} (n-1)! + \binom{k}{2} (n-2)! - \ldots + (-1)^{k} \binom{k}{k} (n-k)! \qquad (13)$$

for every integer n such that $n \ge k$.

Equation 2 shows that Equation 8 holds when k = 0. When Equation 8 holds for a value of k, then it also holds for the next, by Problem 9. Thus, we can increase the value of k successively through all nonnegative integers and conclude that Equation 8 holds for all integers n and k with $0 \leq k \leq n$, as desired.

Setting k equal to n in Equation 8 gives an expression for R(n,n) that simplifies to a famous formula. You can read about this, if you choose, in a postscript to the Team Essay solution on the Math Field Day website.