Tournament Kings

In a <u>tournament</u>, each pair of teams plays one game, which one of the teams wins and the other loses. The number of teams in a tournament can be any integer greater than one.

Represent a tournament by drawing a dot for each team and drawing an arrow from one dot to another when the first team beats the second. For example, Figure 1 shows a tournament with four teams A, B, C, and D, where A beats B and C and loses to D, B beats C and loses to D, and C beats D.

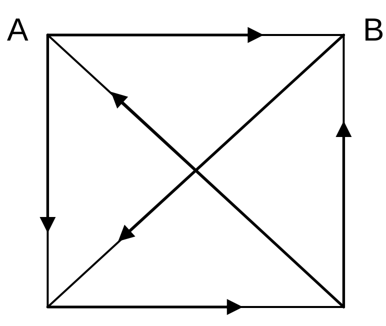
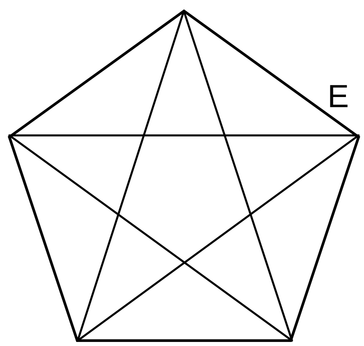


Figure 1



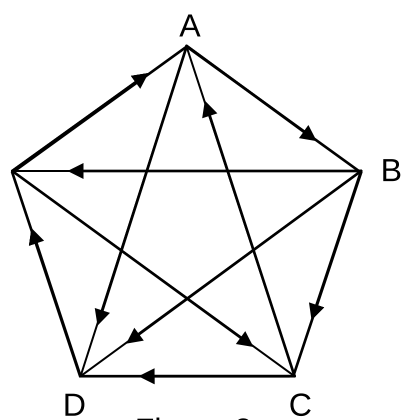


Figure 2

Team X is a <u>king</u> of a tournament if, for every other team Y, X beats Y or a team that beats Y. In the drawing of a tournament, we can get from a king to each other vertex by following one or two arrows.

In Figure 1, A is a king because it beats B and C, and C beats D. B is not a king because it does not beat A or any team that beats A. C is a king because it beats D, and D beats A and B. D is a king because it beats A and B, and A beats C. Thus, three of the four teams in Figure 1 are kings: A, C, and D.

Problem 1. Show that the five-team tournament in Figure 2 has exactly four kings.

Problem 2. Create a six-team tournament where all six teams are kings by starting with Figure 2 and adding a sixth dot F and appropriate arrows. Verify that all six teams A-F are kings.

This essay determines how many teams there can be in a tournament with a specified number of kings. First we consider tournaments where all teams are kings. Problem 2 gives such a tournament with six teams. The next problem shows that we can increase the number of teams in such a tournament by two.

Problem 3. Consider a tournament with k teams where all teams are kings. Enlarge the tournament by adding two new teams X and Y where X beats Y, all k original teams beat X, and Y beats all k original teams. Prove that all k + 2 teams of the enlarged tournament are kings.

Problem 4. For every positive integer $k \geq 3$ except 4, prove that there is a tournament with k teams where all k teams are kings. Make use of Problems 2 and 3.

In contrast with Problem 4, the next two problems show that no tournament with four teams has all teams as kings. On the other hand, Problem 1 shows that a tournament of five teams can have four kings.

Problem 5. Consider a tournament with four teams where no team beats every other team and no team loses to every other team. Prove that the teams can be labeled A, B, C, and D so that Figure 1 shows the tournament.

Problem 6. Prove that no tournament with four teams has all four teams as kings. Make use of Problem 5 and the paragraph before Problem 1.

The next problem shows that every tournament has at least one king.

Problem 7. Let team P win at least as many games as any other team in a tournament. Prove that P is a king.

Hint: Consider the games that any team other than P plays with P and the teams that P beats.

In Figure 1, A and D win two games apiece, and B and C win one. Accordingly, Problem 7 shows that A and D are kings. The next problem shows that C is also a king in Figure 1, since C is the only team that beats D.

Problem 8. Let Q be a team that loses at least once in a tournament. Prove that the teams that beat Q include at least one king.

Hint: If Q loses at least twice, apply Problem 7 to the smaller tournament consisting only of the teams that beat Q and the games they play with each other. Consider separately the case where Q loses only once.

The next problem uses Problem 8 to show that no tournament has exactly two kings.

Problem 9. Suppose that a tournament has at least two kings. Since the two kings play each other, label them Q and R so that Q beats R. Show that Q loses at least once and that Problem 8 gives a king P different from both Q and R.

The next problem shows that we can increase the number of teams in a tournament by any amount without increasing the number of kings.

Problem 10. If there is a tournament with m teams and k kings, prove that there are tournaments with n teams and k kings for all integers n greater than m.

We can now determine the pairs of integers k and n such that there is a tournament with k kings and n teams. For each possible value of k, it suffices by Problem 10 to find the least number of teams in a tournament with k kings. Recall that every tournament has at least two teams, by the first paragraph of the essay.

Problem 11. Prove that there is a tournament with k kings and n teams for exactly those pairs of integers k and n such that $n \geq 2$, $1 \leq k \leq n$, $k \neq 2$, and n and k are not both 4.