# MATH FIELD DAY 2012

## Contestants' Insructions
## Team Essay

1. Your team has *forty* minutes to answer this set of questions.

2. All answers must be justified with complete explanations. Your answers should be clear, grammatically correct, and mathematically precise.

3. Your team may turn in at most one answer to each question. Number the answers and submit them in order, starting each problem on a separate page. You are allowed to skip any question and proceed on to later questions. Write on one side of the paper only, and number the pages you turn in consecutively. Do not turn in the problem packets. Write your school name on each page.

4. Your team may be organized in any way you choose. For example, you may designate one team member to write every answer, or you could designate different team members to write different answers. You may use the chalk or white boards in the room.

5. Only official team members may be in the room during the essay period. Team members may <u>not</u> bring books, calculators, or any other materials into the room.

6. A proctor outside the room will warn you ten minutes before your time is up.

7. Give your answers to the proctor at the end of the period to place in the inner envelope. Please erase all boards in the room and put all scratch paper in the trash can, leaving the room neat.

## Pouring Problems

We have two jugs, one with a capacity of $m$ quarts and the other with a capacity of $n$ quarts, for positive integers $m$ and $n$. We measure amounts of water by pouring from one jug to the other. We can also fill a jug from a nearby pool of water or empty it into the pool. We can tell when a jug is full or empty but, because the jugs are unmarked and oddly shaped, we cannot otherwise tell by looking at a jug how much water it contains. The pouring process is a sequence of steps. It begins with both jugs empty at the start of step 0. Each step has the following parts A and B.

**Part A.** Pour as much water as possible from the $m$-jug to the $n$-jug. This leaves the $m$-jug empty or the $n$-jug full (or both).

**Part B.** If the $m$-jug is empty, fill it from the pool. If the $m$-jug is not empty, then the $n$-jug is full, and empty the $n$-jug into the pool.

We write the symbol $u{:}v$ to show that the $m$-jug currently contains $u$ quarts of water and the $n$-jug currently contains $v$ quarts. We represent each step of the pouring process by two symbols, one before part A and one before part B.

For example, when $m = 3$ and $n = 5$, we pour from a 3-quart jug to a 5-quart jug. Figure 1 shows steps 0 through $m + n = 3 + 5 = 8$ of the pouring process.

| | |
|---|---|
| Step 0. | 0:0, 0:0 |
| Step 1. | 3:0, 0:3 |
| Step 2. | 3:3, 1:5 |
| Step 3. | 1:0, 0:1 |
| Step 4. | 3:1, 0:4 |
| Step 5. | 3:4, 2:5 |
| Step 6. | 2:0, 0:2 |
| Step 7. | 3:2, 0:5 |
| Step 8. | 3:5, 3:5 |

Figure 1

Step 0 starts with 0:0 because both jugs are initially empty. Pouring as much water as possible – none – from the 3-jug to the 5-jug gives the second symbol 0:0 in step 0. Because the 3-jug is empty, we fill it from the pool and start step 1 with 3:0. Emptying the 3-jug into the 5-jug gives 0:3. Because the 3-jug is empty, we fill it from the pool and start step 2 with 3:3. Filling the 5-jug from the 3-jug gives 1:5. Because the 3-jug is not empty and the 5-jug is full, we empty the 5-jug into the pool and start step 3 with 1:0. Emptying the 3-jug into the 5-jug gives 0:1. Because the 3-jug is empty, we fill it from the pool and start step 4 with 3:1. Emptying the 3-jug into the 5-jug gives 0:4. Because the 3-jug is empty, we fill it from the pool and start step 5 with 3:4. Filling the 5-jug from the 3-jug gives 2:5. Because the 3-jug is not empty and the 5-jug is full, we empty the 5-jug into the pool and start step 6 with 2:0. Emptying the 3-jug into the 5-jug gives 0:2. Because the 3-jug is empty, we fill it from the pool and start step 7 with 3:2. Emptying the 3-jug into the 5-jug gives 0:5. Because the 3-jug is empty, we fill it from the pool and start step 8 with 3:5. Pouring as much water as possible–none–from the 3-jug to the 5-jug gives the second symbol 3:5 in step 8.

> **Problem 1.** *Show steps 0 through $m+n$ of the pouring process for $m=5$ and $n=8$. Represent each step by two symbols u:v, as in Figure 1. You do not need to describe the steps verbally as in the previous paragraph.*

Let $y_i$ be the total number of quarts in the two jugs during step $i$ of the pouring process. That is, $y_i$ is the sum $u+v$ for the two symbols $u:v$ representing that step. For example, when $m=3$ and $n=5$, Figure 1 shows that $y_2=6$, since step 2 has the symbols 3:3 and 1:5 and $3+3=6=1+5$. That is, there are 6 quarts of water in the two jugs combined during step 2.

Consider the sequence

$$y_1, y_2, \ldots, y_{m+n} \tag{1}$$

of numbers of quarts in steps 1 through $m+n$. Steps 1 through 8 of Figure 1 show that (1) is the sequence

$$3, 6, 1, 4, 7, 2, 5, 8 \tag{2}$$

when $m=3$ and $n=5$.

> **Problem 2.** *Find the sequence (1) for $m=5$ and $n=8$, as in (2). Use your answer to Problem 1.*

Problem 5 shows that (1) consists of the integers 1 through $m+n$ in some order whenever $m$ and $n$ have no common factor greater than 1. The sequence (2) illustrates this result for $m=3$ and $n=5$, and your answer to Problem 2 should illustrate it for $m=5$ and $n=8$.

When $m$ and $n$ have no common factor greater that 1, the fraction $\frac{m}{n}$ is in lowest terms. The next claim describes the fractions equal to $\frac{m}{n}$.

**Claim I.** Let $m$ and $n$ be positive integers that have no common factor greater than 1. If $a$ and $b$ are positive integers such that $\frac{a}{b}=\frac{m}{n}$, then $a=mt$ and $b=nt$ for a positive integer $t$.

Claim I says that we get $\frac{m}{n}$ by canceling factors of $t$ in the numerator or denominator of $\frac{a}{b}$. Problems 8 through 10 prove Claim I. We use Claim I until then to study the pouring process.

**Problem 3.** *Assume that $m$ and $n$ have no common factor greater than 1. Let $r$ and $s$ be positive integers such that $y_r = y_s$ and $r < s$.*

*(a) Let $h$ be the number of times we fill the $m$-jug from the pool and let $k$ be the number of times we empty the $n$-jug into the pool from the start of step $r$ to the start of step $s$ (that is, when we perform part B in steps $r$ through $s-1$). Deduce from Claim 1 that $h = nt$ and $k = mt$ for a positive integer $t$.*

*(b) Conclude that $s - r \geq m + n$.*

**Problem 4.** *Why is $y_j$ an integer from 1 through $m+n$ for every positive integer $j$?*

**Problem 5.** *Assume that $m$ and $n$ have no common factor greater than 1. Prove that the sequence $y_1, ..., y_{m+n}$ in (1) consists of the integers 1 through $m+n$ in some order. Use Problems 3 and 4.*

Problem 5 shows that the pouring process gives every whole number of quarts from 1 through $m+n$ when $m$ and $n$ have no common factor greater than 1.

**Problem 6.** *Assume that $m$ and $n$ have no common factor greater than 1. Let $s$ be a positive integer such that $y_s = m+n$.*

*(a) Let $h$ be the number of times we fill the $m$-jug from the pool and let $k$ be the number of times we empty the $n$-jug into the pool from the start of step 0 to the start of step $s$ (that is, when we perform part B in steps 0 through $s-1$). Deduce from Claim I that $h = nt+1$ and $k = mt-1$ for a positive integer $t$.*

*(b) Conclude from (a) that $s \geq m+n$.*

**Problem 7.** *Assume that $m$ and $n$ have no common factors greater than 1.*

*(a) Deduce from Problems 5 and 6 that $y_{m+n} = m+n$.*

*(b) Why does $y_j = y_{m+n+j}$ for every positive integer $j$?*

For $m = 3$ and $n = 5$, Problem 7(a) says that $y_8 = 8$, as the last entry in (2) shows. Likewise, the last entry in your answer to Problem 2 should illustrate Problem 7(a) for $m = 5$ and $n = 8$. Problem 7 (b) says that the sequence $y_j$ repeats every $m+n$ steps when $y_0$ is set aside.

The three remaining problems justify Claim I. They do not rely on factoring integers into primes, and neither should your answers.

**Problem 8.** *Let $a$, $b$, $c$, and $d$ be positive integers such that*

$$\frac{a}{b} = \frac{c}{d}$$

*Let $t$ be an integer.*

*(a) If $b = dt$, prove that $a = ct$.*

*(b) If $b - dt > 0$, prove that $a - ct > 0$ and*

$$\frac{a - ct}{b - dt} = \frac{a}{b}.$$

Let $b$ and $d$ be positive integers. Let $t$ be the largest integer such that $b - dt \geq 0$. Since $b - d(t + 1) < 0$, we have

$$0 \leq b - dt < d. \tag{3}$$

For example, $b = 25$ and $d = 7$ give $t = 3$ and

$$0 \leq 25 - 7(3) < 7.$$

**Problem 9.** *Let $a$ and $b$ be positive integers. Let $c$ and $d$ be positive integers such that*

$$\frac{a}{b} = \frac{c}{d}$$

*and $d$ is as small as possible. Deduce from (3) and Problem 8 that $a = ct$ and $b = dt$ for a positive integer $t$.*

**Problem 10.** *Let $m$ and $n$ be positive integers that have no common factor greater than 1. Let $c$ and $d$ be positive integers such that*

$$\frac{c}{d} = \frac{m}{n}$$

*and $d$ is as small as possibe. Deduce from Problem 9 that $c = m$ and $d = n$.*

Let

$$\frac{a}{b} = \frac{m}{n}$$

for positive integers $a$, $b$, $m$, and $n$ such that $m$ and $n$ have no common factor greater than 1. By problem 10, we can take $c = m$ and $d = n$ in Problem 9 and see that $a = mt$ and $b = nt$ for a positive integer $t$. This proves Claim I.