MATH FIELD DAY 2013

Contestants' Instructions
Team Essay

1. Your team has *forty* minutes to answer this set of questions.

2. All answers must be justified with complete explanations. Your answers should be clear, grammatically correct, and mathematically precise.

3. Your team may turn in at most one answer to each question. Number the answers and submit them in order, starting each problem on a separate page. You are allowed to skip any question and proceed on to later questions. Write on one side of the paper only, and number the pages you turn in consecutively. Do not turn in the problem packets. Write your school name on each page.

4. Your team may be organized in any way you choose. For example, you may designate one team member to write every answer, or you could designate different team members to write different answers. You may also use the chalk board.

5. Only official team members may be in the room during the essay period. Team members may <u>not</u> bring books, calculators, or any other materials into the room.

6. A proctor outside the room will warn you ten minutes before your time is up.

7. Give your answers to the proctor at the end of the period to place in the inner envelope. Please erase all chalk boards in the room and put all scratch paper in the trash can, leaving the room neat.

Triangle-Free Graphs

A graph consists of finitely many points (called vertices) and a set of segments (called edges) that have the vertices as endpoints. For example, Figure 1 shows a graph with six vertices A, B, C, D, E, F and six edges AD, AE, BE, BF, CF, EF.

It does not matter how the vertices of a graph are positioned. For example, Figure 2 shows the same graph as Figure 1. The extra crossings of the edges in Figure 2 are not significant.

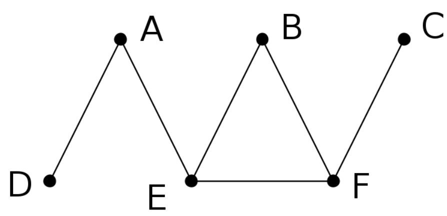

Figure 1

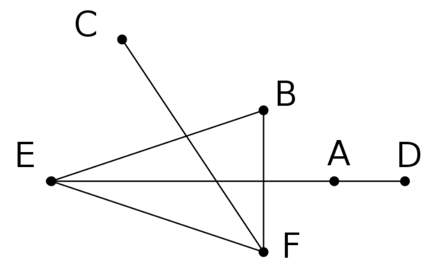

Figure 2

A graph contains triangle XYZ if X, Y, and Z are three vertices of G and G has the three edges XY, YZ, and XZ joining X, Y, and Z in pairs. For instance, the graph in Figure 1 (or Figure 2) contains triangle BEF.

A graph is triangle-free if it does not contain any triangles. For example, Figure 3 shows a triangle-free graph with 5 vertices and 4 edges. Adding any edge to Figure 3 gives a triangle; for instance, adding edge BC gives triangle BCD, adding edge AC gives triangle ACD, and so on.

Figure 4 shows a triangle-free graph with 5 vertices and 5 edges. Adding any edge gives a triangle; for instance, adding edge AD gives triangle ADE, adding edge AC gives triangle ACB, and so on.

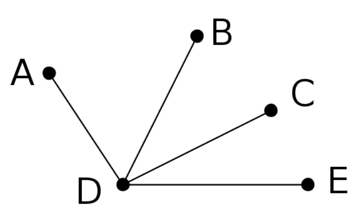

Figure 3

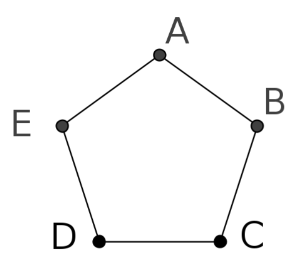

Figure 4

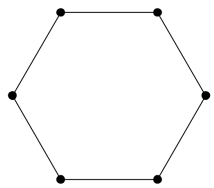

Figure 5

Problem 1. The hexagon in Figure 5 is a triangle-free graph with 6 vertices and 6 edges. Show how to add 3 more edges and still have a triangle-free graph. No explanation is needed.

Our goal is to find the largest possible number of edges that a triangle-free graph with n vertices can have.

For each positive integer n, we specify a graph T_n as follows: T_n has n vertices divided into two sets L and R, where L and R have the same number of vertices when n is even, and L has one more vertex than R when n is odd. The edges of T_n are all the segments with one endpoint in L and the other in R.

We draw T_n so that the vertices in L form a column on the left and the vertices in R form a column on the right. We draw all edges that have one endpoint on the left and one on the right.

For example, Figure 6 shows T_6. There are the same number of vertices on the left and right because 6 is even.

Figure 7 shows T_5. there is one more vertex on the left than the right because 5 is odd.

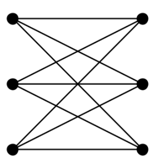

Figure 6

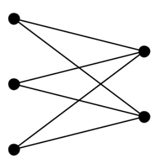

Figure 7

Problem 2. Draw T_8. How many edges does it have?

We are going to prove the following result:

Mantel's Theorem. For any positive integer n, T_n is a triangle-free graph with n vertices. T_n has more edges than any other triangle-free graph with n vertices.

Mantel's Theorem says that T_n is the one triangle-free graph with n vertices that has the most edges. For example, since T_5 has 6 edges (Figure 7), every other triangle-free graph with 5 vertices has fewer than 6 edges (such as Figures 3 and 4, for example).

Problem 3. Use Mantel's Theorem to draw the triangle-free graph with 7 vertices that has the greatest possible number of edges. How many edges does it have?

Problem 4. Deduce from Mantel's Theorem that the greatest possible number of edges in a triangle-free graph with n vertices is $n^2/4$ if n is even and $(n^2 - 1)/4$ if n is odd.

Problem 5. Draw a copy of your answer to Problem 1. Label three of the vertices L and the other three vertices R so that the edges of the graph are all the segments with one vertex labeled L and labeled R. (This shows that the graph is T_6.)

The rest of this essay proves Mantel's Theorem. Do not use Mantel's Theorem to do any of the remaining problems.

Problem 6. Show that T_n is triangle-free for every positive integer n.

Let a graph G have an edge XY. Then $G - \{X, Y\}$ is the graph we get from G by deleting vertices X and Y, edge XY, and all other edges having X or Y as an endpoint. For example, if G is the graph in Figure 8, $G - \{A, B\}$ is the graph in Figure 9.

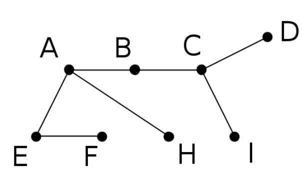

Figure 8

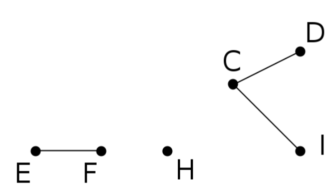

Figure 9

Problem 7. For the graph G in Figure 10, draw $G - \{C, H\}$.

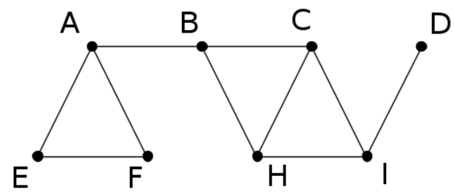

Figure 10

We study triangle-free graphs by relating those with $n+2$ vertices to those with n vertices for each positive integer n. If T_6 has edge AB (Figure 11), then $T_6 - \{A, B\}$ is T_4 (Figure 12), and T_6 has 5 more edges than T_4. This illustrates the next problem for $n = 4$.

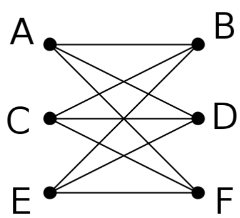

Figure 11

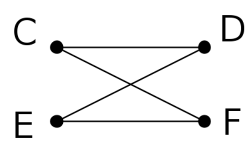

Figure 12

Problem 8. Let AB be an edge in T_{n+2} for a positive integer n. Show that $T_{n+2} - \{A, B\}$ is T_n and that T_{n+2} has $n + 1$ more edges than T_n.

If we replace T_{n+2} in Problem 8 by any triangle-free graph G with $n + 2$ vertices, G may not have $n + 1$ more edges than $G - \{A, B\}$ (in fact, G may not even have $n + 1$ edges), but G will have at most $n + 1$ more edges than $G - \{A, B\}$. For example, the triangle-free graph G in Figure 8 has $8 = n + 2$ vertices for $n = 6$, and it has 7 edges. Figure 9 shows $G - \{A, B\}$, which has $n = 6$ vertices and 3 edges. G has 4 more edges than $G - \{A, B\}$, and 4 is less than $n + 1 = 7$. This illustrates the next problem.

Problem 9. Let G be a triangle-free graph with $n + 2$ vertices for a positive integer n, and let AB be an edge of G. Show that G has at most $n+1$ more edges than $G - \{A, B\}$.

The next problem shows that the conditions in Problem 8 distinguish T_{n+2} from other triangle-free graphs with $n + 2$ vertices.

Problem 10. Let G be a triangle-free graph with $n+2$ vertices for a positive integer n, and let AB be an edge of G. If $G - \{A, B\}$ is T_n and G has $n+1$ more edges than T_n, prove that G is T_{n+2}.

Consider the following statement about a positive integer n:

Statement I. Every triangle-free graph with n vertices that doesn't equal T_n has fewer edges than T_n.

We want to prove that this statement is true for every positive integer n.

Problem 11. Verify directly that Statement I is true for $n = 1$ and $n = 2$.

Problem 12. Let k be a positive integer such that Statement I is true for $n = k$. Use Problems 8-10 to show that Statement I is true for $n = k + 2$.

Problem 13. Explain why Mantel's Theorem follows from Problems 6, 11, and 12.

Paul Turàn proved a famous generalization of Mantel's Theorem in 1941.