Answers to 2004 Team Essay

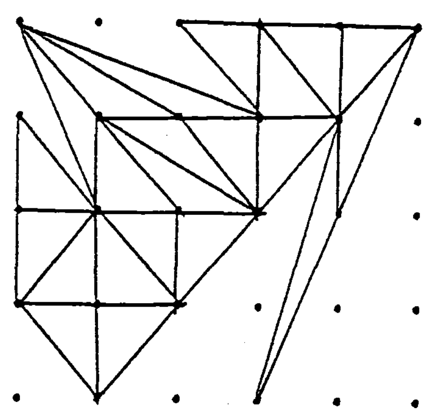
1. There are $i = 4$ lattice points inside P and $s = 16$ lattice points on the sides of P. Pick's Theorem states that P has area $i + (s/2) - 1 = 4 + (16/2) - 1 = 11$.

2. For any basic triangle T, we have $i = 0$ because there are no lattice points inside T. We have $s = 3$ because the vertices are the only lattice points on the sides of T. Thus, we have $V(T) = i + (s/2) - 1 = 0 + (3/2) - 1 = 1/2$.

3. When T has one side in common with P, the vertex of T not on this side lies on the sides of Q but not P (Figures 6 and 7). The other two vertices of T lie on the sides of both P and Q. Because T is a basic triangle, its vertices are the only lattice points on or inside of T. Thus, replacing P with Q leaves i unchanged and increases s by 1, and so it increases $i + (s/2) - 1$ by $\frac{1}{2}$. This shows that $V(Q) = V(P) + \frac{1}{2}$.

When T shares two sides with P, their common vertex lies on the sides of P but inside Q (Figures 8 and 9). The other two vertices of T lie on the sides of both P and Q. Because T is a basic triangle, its vertices are the only lattice points on or inside of T. Thus, replacing P with Q increases i by 1 and decreases s by 1, and it increases $i + (s/2) - 1$ by $1 - \frac{1}{2} = \frac{1}{2}$. This shows that $V(Q) = V(P) + \frac{1}{2}$ in this case as well.

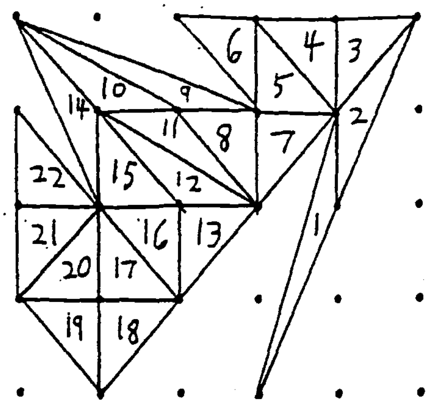
4.



5. Moving outwards from one of them, we can number the basic triangles $T_1, \ldots, T_n$ so that $T_1, \ldots, T_k$ form a polygon P_k for each integer k from 1 through n. (For example, the figure on the next page shows one such way to number the triangles in the answer to Problem 4.) If $k < n$, then T_{k+1} has one or two sides in common with P_k, and so we have the relation

$$V(P_{k+1}) = V(P_k) + \tfrac{1}{2}$$

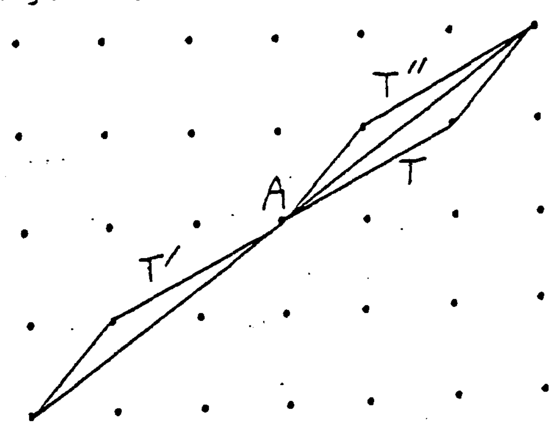
(by Problem 3).



Since $P_1 = T_1$ is a basic triangle, Problem 2 shows that $V(P_1) = \frac{1}{2}$. Applying the last paragraph repeatedly gives $V(P_2) = V(P_1) + \frac{1}{2} = 2(\frac{1}{2})$, $V(P_3) = V(P_2) + \frac{1}{2} = 3(\frac{1}{2})$, and so on through $V(P_n) = n(\frac{1}{2})$. Since $P_n = P$, we have $V(P) = n/2$.

6. By Claim 1, we can divide any lattice polygon P into basic triangles whose interiors do not overlap. If there are n of these triangles, then P has area n/2 (since each basic triangle has area $\frac{1}{2}$, by Claim 2, and since the triangles don't overlap except on their edges). Since V(P) also equals n/2 (by Problem 5), it equals the area of P. Thus, the area of P is $i + (s/2) - 1$, and Pick's Theorem holds.

7. (a) Rotating the plane 180° about a lattice point interchanges all lattice points in the plane among themselves. Thus, rotating the basic triangle T 180° about one of its vertices A gives a basic triangle T'.



Translate the plane so that A translates to another vertex of T. This translation takes T' to a triangle T" that forms a parallelogram B with T. Because the translation interchanges lattice points among themselves, the fact that T' is a basic triangle implies that T" is also.

Because T and T" are basic triangles, the parallelogram B they form is a basic parallelogram. T and T" are congruent because rotations and translations preserve lengths and angles.

(b) Because T and T" are congruent, they have the same area. Thus, the area of the parallelogram B formed by T and T" is twice the area of T. Since B is a basic parallelogram, it has area 1, by Claim 3. Thus, T has area $\frac{1}{2}$.

8. (a) Let S be the square region enclosed by the lines $x = 1 + L$, $x = n - L$, $y = 1 + L$, and $y = n - L$. For any point (a,b) in S, we have

$$1 + L \leq a \leq n - L \quad \text{and} \quad 1 + L \leq b \leq n - L. \tag{1}$$

Because the translations of B tile the plane, one of them contains (a,b). Let (j,k) be its base point. Because any two points of B are at most L units apart, the distance from (a,b) to (j,k) is at most L. Thus we have $|j - a| \leq L$ and $|k - b| \leq L$. Together with the inequalities in (1), this shows that $1 \leq j \leq n$ and $1 \leq k \leq n$. In short, every point (a,b) of S lies in a translate of B having base point (j,k) for $1 \leq j \leq n$ and $1 \leq k \leq n$.

(b) By part (a), the square region S is covered by translates of B having base points (j,k) for $1 \leq j \leq n$ and $1 \leq k \leq n$. There are n^2 of these translates, and they each have area A, and so the total area they cover is n^2A. This is at least the area of S because the translates cover S. Because S is a square with sides of length $n - L - (1 + L) = n - 2L - 1$, it has area $(n - 2L - 1)^2$. Thus, we have $n^2A \geq (n - 2L - 1)^2$.

9. (a) Consider any translate of B that has base point (j,k) for

$$1 \leq j \leq n \quad \text{and} \quad 1 \leq k \leq n. \tag{2}$$

Let (c,d) be any point of this translate. Because any two points of B lie at most L units apart, the distance from (j,k) to (c,d) is at most L. Thus, we have $|c - j| \leq L$ and $|d - k| \leq L$. Together with the inequalities in (2), this shows that $1 - L \leq c \leq n + L$ and $1 - L \leq d \leq n + L$. In short, all translates of B whose base points (j,k) satisfy (2) lie in the square region T enclosed by the lines $x = 1 - L$, $x = n + L$, $y = 1 - L$, and $y = n + L$.

(b) The square T has sides of length $n + L - (1 - L) = n + 2L - 1$, and so it has area $(n + 2L - 1)^2$. By part (a), this square contains n^2 translates of B, which each have area A. Because these translates overlap only along their edges, they cover a total area of n^2A. This is at most the area of the square T containing the translates. Thus, we have $n^2A \leq (n + 2L - 1)^2$.

10. Combining Problems 8(b) and 9(b) shows that

$$(n - 2L - 1)^2 \leq n^2A \leq (n + 2L - 1)^2.$$

Collecting powers of n in the first and last expressions gives

$$n^2 - 2n(2L+1) + (2L+1)^2 \leq n^2A \leq n^2 + 2n(2L-1) + (2L-1)^2.$$

Dividing by n^2 shows that

$$1 - \frac{4L+2}{n} + \frac{4L^2+4L+1}{n^2} \leq A \leq 1 + \frac{4L-2}{n} + \frac{4L^2-4L+1}{n^2}.$$

11. If it were true that $A < 1$, we could choose a positive integer n so large that

$$\frac{4L+2}{n} < 1 - A.$$

Then we would have

$$A < 1 - \frac{4L+2}{n} < 1 - \frac{4L+2}{n} + \frac{4L^2+4L+1}{n^2},$$

which would contradict Problem 10.

If it were true that $A > 1$, we could choose a positive integer n so large that

$$\frac{4L-2}{n} + \frac{4L^2-4L+1}{n^2} < A - 1.$$

Then we would have

$$1 + \frac{4L-2}{n} + \frac{4L^2-4L+1}{n^2} < A,$$

which would contradict Problem 10.

Because neither of the inequalities $A < 1$ or $A > 1$ can hold, we must have $A = 1$.