Solutions to the Team Essay on Trading Chips

1.

R	W	B	Trade-in
3	9	1	RW
2	8	3	WB
4	7	2	RW
3	6	4	WB
5	5	3	

2.

R	W	B	Trade-in
10	5	0	RW
9	4	2	RW
8	3	4	RB
7	5	3	RW
6	4	5	

3. Let $x < y < z$ be the numbers of chips of the three colors before the basic step. After the basic step, the numbers are $x + 2$, $y – 1$, and $z – 1$, respectively. Since $y < z$, we have $y – 1 < z – 1$. The assumption that $z \geq x + 3$ means that $z – 1 \geq x + 2$. It follows that $z – 1 > x + 2$, since we are assuming that no two colors have the same number of chips after the basic step. In short, $z – 1$ is greater than both $y – 1$ and $x + 2$. Thus, the most common color of chip is unchanged by the basic step, and the number of chips of that color has decreased by 1.

4. Repeat the basic step as long as the numbers of chips of the three colors are all different and the largest is at least 3 more than the smallest. By Problem 3, the most common color stays the same and decreases by 1 each step. This process ends because the number of chips of the most common color is a positive integer and cannot decrease forever.

   When the process ends, one possibility is that two or more colors have the same number of chips, and we have reached (A). The other possibility is the numbers of chips of the three colors are all different and the largest is at most 2 more than the smallest. It follows that these numbers are three consecutive integers, since the only way to take three different integers from k through $k + 2$ is to take k, $k + 1$, and $k + 2$.

5.

R	W	B	Trade-in
3	9	1	RW
2	8	3	WB
4	7	2	RW
3	6	4	WB
5	5	3	RW
4	4	5	RW
3	3	7	RW
2	2	9	RW
1	1	11	RW
0	0	13	

6. Assume that the total number of chips is not a multiple of 3. This remains true after any sequence of trades, since trades do not change the total number of chips. Thus, no sequence of trades can produce consecutive integers for the numbers of chips of the three colors: the sum of three consecutive integers is a multiple of 3, since $k + (k + 1) + (k + 2) = 3k + 3 = 3(k + 1)$.

   The standard procedure always reaches (A) or (B), by Problem 4. It cannot reach (B) when the total number of chips is not a multiple of 3, by the last paragraph, and so it reaches (A). Thus, we can get the same number of chips of two colors. We can then make all the chips the same color, as before Problem 5.

7. Consider the difference $x – y$, where x and y are the numbers of chips of two colors. A trade either decreases both x and y by 1 or decreases just one of them by 1 and increases the other by 2. Decreasing both x and y by 1 changes the difference by 0, since $(x – 1) – (y – 1) = x – y$. Decreasing x by 1 and increasing y by 2 changes the difference by -3, since $(x – 1) – (y + 2) = x – y – 3$. Increasing x by 2 and decreasing y by 1 changes the difference by 3, since $(x + 2) – (y – 1) = x – y + 3$.

8. When all the chips are one color, the difference of the numbers of chips of the other two colors is $0 – 0 = 0$. When the numbers of chips of the three colors are consecutive integers, each difference in (1) is ± 1 or ± 2. If we take a difference in (1) after one sequence of trades and subtract the same difference after another sequence of trades, we get a multiple of 3, as noted after Problem 7. Thus, if one sequence of trades reaches (B), no sequence of trades can make all the chips one color, since subtracting 0 from ±1 or ± 2 does not give a multiple of 3.

9. The total number n of chips is unchanged by trades. Consider any sequence of trades that makes all the chips one color. The two differences in (1) that involve this color and one of the others are $\pm (n – 0) = \pm n$ after the sequence of trades, and the difference involving the other two colors is $0 – 0 = 0$. Since n is not a multiple of 3 but 0 is, and

since a sequence of trades changes each difference by a multiple of 3 (as after Problem 7), exactly one of the differences in (1) is a multiple of 3 before any trades, and the color not involved in this difference is the color that the sequence of trades made all the chips.

Problem 6 shows that there is a sequence of trades making all the chips one color. Thus, by the previous paragraph, exactly one of the differences in (1) is a multiple of 3 before any trades, and the color not involved in this difference is the one color that all the chips can be made.