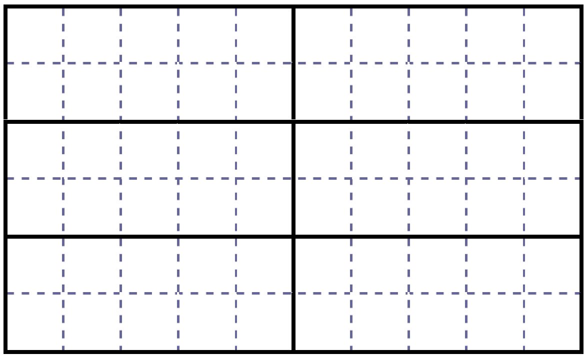
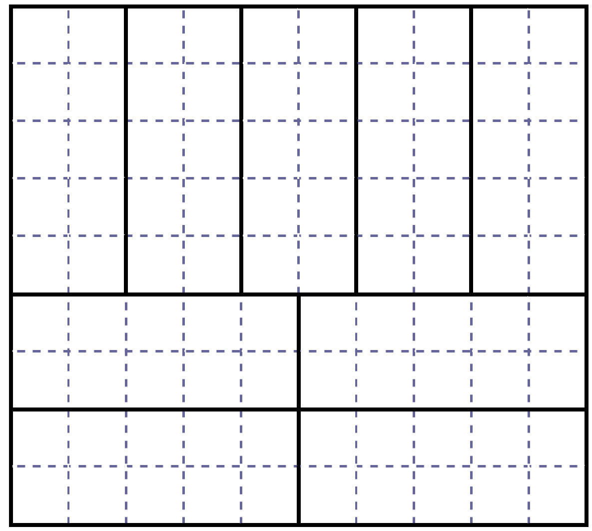
Team Essay 2007 Solutions

1.



2.



3. Draw a line cs units below the top edge of an $m \times n$ rectangle. This line divides the rectangle into a $cs \times n$ rectangle and a $dt \times n$ rectangle since $m = cs + dt$. Since cs is a multiple of c and n is a multiple of d, the observation after Problem 2 shows that the $cs \times n$ rectangle has a (c,d) tiling where all the tiles have orientation $c \times d$. Since dt is a multiple of d and n is a multiple of c, the observation after Problem 2 shows that the $dt \times n$ rectangle has a (c,d) tiling where all the tiles have orientation $d \times c$. Combining these two tilings gives a (c,d) tiling of the $m \times n$ rectangle, where the tiles of orientation $c \times d$ form a $cs \times n$ rectangle and the tiles of orientation $d \times c$ form a $dt \times n$ rectangle.

4. Consider the tiles along the left edge of the rectangle. Let s of them have orientation $c \times d$, and let t have orientation $d \times c$, where s and t are nonnegative integers. These tiles divide the edge into s segments of length c and t segments of length d. Since the total length m of the edge is the sum of the lengths of these segments, we have $m = cs + dt$.

5. Say red is color of column 1 and green the color of column c. Because all c colors repeat cyclically, there is at most one more red column than green. When n is not a multiple of c, there cannot be the same number of columns of all c colors, and there are more red than green columns. Thus, there is exactly one more red column than green. Since every column has m unit squares, the number x of red unit squares is m more than the number y of green unit squares, and we have $x - y = m$.

6. (a) Pick any color, say, red. If a tile of orientation $c \times d$ overlaps k red columns, ck unit squares of the tile are red, since each column of the tile has c unit squares. Thus, the number of red unit squares in the tile is the multiple ck of c.

(b) Any c consecutive columns of the rectangle have exactly one column of each color, since the c colors of the columns repeat cyclically. A tile of orientation $d \times c$ has c consecutive columns, and so it has exactly one column of each color. Since each column of the tile has d unit squares, the tile has exactly d unit squares of each color.

7. Say red is the most common color. In each tile of orientation $c \times d$, the number of red unit squares is a multiple of c (by problem 6(a)). Totaling these numbers over all tiles of orientation $c \times d$ gives cu for an integer u. The w tiles of orientation $d \times c$ contain dw red unit squares because each one contains exactly d (by Problem 6(b)). Since every unit square lies in exactly one tile of orientation $c \times d$ or $d \times c$, we get the total number x of red unit squares by taking the number cu in tiles of orientation $c \times d$ and adding the number dw in tiles of orientation $d \times c$. Thus, we have $x = cu + dw$, where u is an integer. Applying the same argument to the least common color shows that $y = cv + dw$ for an integer v.

8. If n is a multiple of c, we're done. If not, we have

$$\begin{aligned} m &= x - y \text{ (by Problem 5)} \\ &= (cu + dw) - (cv + dw) \text{ (by Problem 7)} \\ &= cu + dw - cv - dw \\ &= cu - cv \\ &= c(u - v), \end{aligned}$$

and m is a multiple of c (since $u - v$ is an integer). Thus, either n or m is a multiple of c.

9. If condition (i) of the Rectangular Tiling Theorem holds, the rectangle has a (c,d) tiling, by the observation after Problem 2. If condition (ii) holds, the rectangle has a (c,d) tiling, by Problem 3.

Conversely, suppose that an $m \times n$ rectangle has a (c,d) tiling. By Problem 8, m or n is a multiple of c. Interchanging c and d shows that m or n is a multiple of d. If m or n is a multiple of c and the other is a multiple of d, condition (i) holds. Thus, we can assume that either m or n is a multiple of both c and d.

Since m and n are interchangeable, we can assume that n is a multiple of both c and d. By Problem 4, we have $m = cs + dt$ for nonnegative integers s and t. If $s = 0$, then $m = dt$ is a multiple of d, and condition (i) holds because n is a multiple of c. If $t = 0$, then $m = cs$ is a multiple of c, and condition (i) holds because n is a multiple of d. Thus, we can assume that $m = cs + dt$ for positive integers s and t. Then condition (ii) holds because n is a multiple of both c and d.