2008 Team Essay Answers

Fibonacci Numbers and Greatest Common Divisors

1. $f_{13} = f_{11} + f_{12} = 89 + 144 = 233.$

   $f_{14} = f_{12} + f_{13} = 144 + 233 = 377.$

   $f_{15} = f_{13} + f_{14} = 233 + 377 = 610.$

   $f_{16} = f_{14} + f_{15} = 377 + 610 = 987.$

2. Equation 3 checks for b = 9 and c = 6, since

   $f_{10}f_{15} – f_9f_{16} = 55(610) - 34(987) = 33550 – 33558 = -8 = (-1)^9 f_6$.

3. $(f_{63}, f_{98}) = f_{(63,98)} = f_7 = 13.$

4. Since a > b, a – b is positive, and we can substitute it for c in (3). This gives

$$f_{b+1}f_a – f_bf_{a+1} = (-1)^b \; f_{a-b}.$$

Substituting ds for a and dt for f_b shows that $(-1)^b f_{a-b}$ equals

$$f_{b+1}ds – dtf_{a+1} = d(s f_{b+1} – t f_{a+1}).$$

Multiplying by ±1 shows that f_{a-b} is d times an integer.

5. When a ≥ b + 2, a – b – 1 is positive, and we can substitute it for c in (2). This gives

$$f_a = f_bf_{a-b-1} + f_{b+1}f_{a-b}.$$

Substituting dt for f_b and du for f_{a-b} shows that

$$f_a = dtf_{a-b-1} + f_{b+1}du = d(tf_{a-b-1} + uf_{b+1}du),$$

and so f_a is a multiple of d.

When $a = b + 1$, we have $f_{a-b} = f_1 = 1$. If this is a multiple of a positive integer d, then d equals 1, and f_a is a multiple of d (since every integer is a multiple of 1).

6. If $a > b$, then $c = a – b$ and $d = b$ are positive integers such that $c + d = a < a + b$. Substituting for c and d in the second sentence of the problem shows that

$$(f_{a-b},f_b) = f_{(a-b,b)} .$$

Together with Equations 5 and 6, this shows that

$$(f_a,f_b) = (f_{a-b},f_b) = f_{(a-b,b)} = f_{(a,b)} ,$$

as desired.

If $a < b$, switching a and b in the last paragraph shows that $(f_b,f_a) = f_{(b,a)}$, which is the same as (7).

If $a = b$, we have

$$(f_a,f_b) = (f_a,f_a) = f_a = f_{(a,a)} = f_{(a,b)} ,$$

since $(m,m) = m$ for every positive integer m,

7. Adding the equations

$$f_{b+k+1} = f_bf_k + f_{b+1}f_{k+1}$$

and

$$f_{b+k+2} = f_bf_{k+1} + f_{b+1}f_{k+2}$$

gives

$$f_{b+k+1} + f_{b+k+2} = f_b(f_k +f_{k+1}) + f_{b+1}(f_{k+1} + f_{k+2}).$$

It follows that

$$f_{b+k+3} = f_{b+k+2} + f_{b+1}f_{k+3},$$

since taking n = b + k + 1 in (1) shows that

$$f_{b+k+3} = f_{b+k+1} + f_{b+k+2},$$

taking n = k in (1) shows that

$$f_{k+2} = f_k + f_{k+1},$$

and taking n = k + 1 in (1) shows that

$$f_{k+3} = f_{k+1} + f_{k+2}.$$

8. Swtting c = 1 in (2) gives the equation

$$f_{b+2} = f_b f_1 + f_{b+1} f_2.$$

This equation holds because f_1 and f_2 equal 1 and Equation 1 shows that $f_{b+2} = f_b + f_{b+1}$.

Setting c = 2 in (2) gives the equation

$$f_{b+3} = f_b f_2 + f_{b+1} f_3.$$

This equation holds because $f_2 = 1$, $f_3 = 2$, and

$$\begin{aligned} f_b + 2f_{b+1} &= f_b + f_{b+1} + f_{b+1} \\ &= f_{b+2} + f_{b+1} \qquad \text{(by (1) with n = b)} \\ &= f_{b+3} \qquad \text{(by (1) with n = b+ 1)} \end{aligned}$$

9. Taking n = k and n = k -1 in (1) shows that

$$f_{k+2} f_k - f_{k+1}^2 = (f_k + f_{k+1}) f_k - f_{k+1}(f_{k-1} + f_k).$$

Multiplying out the right-hand side and canceling $\pm f_{k+1} f_k$ gives $f_k^2 - f_{k+1} f_{k-1}$, which is -1 times $f_{k+1} f_{k-1} - f_k^2$.

10. Setting b = 2 in (8) gives the equation $f_3 f_1 - f_2^2 = 1$, which holds because the left-hand side has value $2(1) - 1^2 = 1$.

If (8) holds when b is an integer k, we have the equation

$$f_{k+1}f_{k-1} – f_k^2 = (-1)^k.$$

Replacing k by k+ 1 multiplies the left side of this equation by -1 (by Problem 9) and also multiplies the right side by -1 (since $(-1)^{k+1} = (-1)(-1)^k$). Thus, (8) still holds when b is k + 1.

The second-to-last paragraph shows that (8) holds for b = 2. Then (8) also holds for b = 3, by the last paragraph. This implies that (8) holds for b = 4, by the last paragraph. Continuing in this way shows that (8) holds for every integer b ≥ 2.

11. When b ≥ 2, Equations 9 and 2 show that $f_{b+1}f_{b+c} – f_bf_{b+c+1} =$

$f_{b+1}(f_{b-1}f_c + f_bf_{c+1}) – f_b(f_bf_c + f_{b+1}f_{c+1})$.

Multiplying out the right side and canceling $\pm f_{b+1}f_bf_{c+1}$ gives

$$f_{b+1}f_{b-1}f_c – f_b^2\ f_c.$$

Factoring out f_c gives

$$(f_{b+1}f_{b-1} – f_b^2)\ f_c,$$

which equals $(-1)^bf_c$, by (8).

When b = 1, (3) says that

$$f_2f_{c+1} – f_1f_{c+2} = -f_c\ .$$

This holds because f_1 and f_2 equal 1 and we can substitute $f_c + f_{c+1}$ for f_{c+2} (by (1) with n = c).