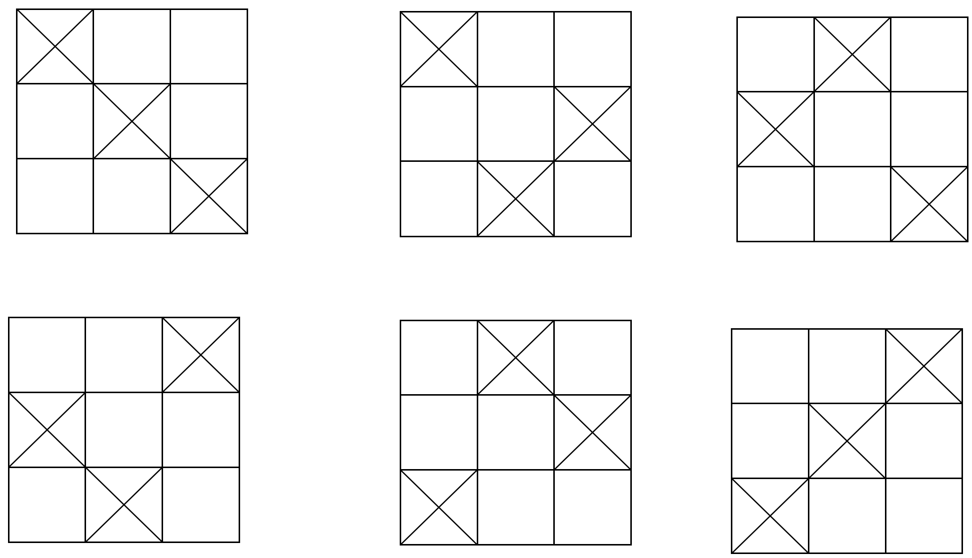
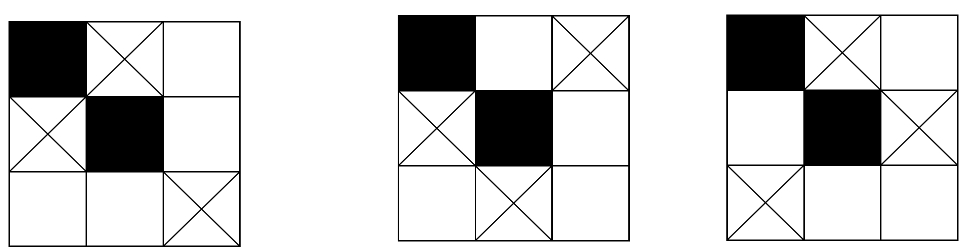
Solutions and Postscript to the 2009 Team Essay

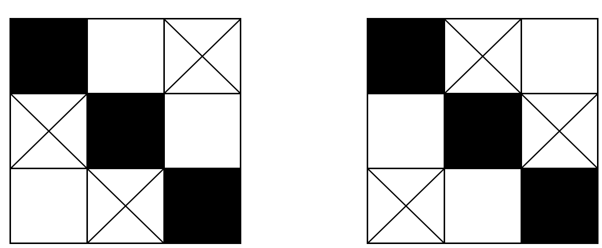
1. R(3,0) = 6:



R(3,2) = 3:



R(3,3) = 2:



2. Divide the R(n,1) ways to place rooks on an (n,1) board into two groups, depending on whether or not there is a rook in row 2 of column 2. If there is, position other rooks so that exactly one of them lies in each of the remaining n – 1 rows and n – 1 columns and there is no rook in row 1 of column 1. This is equivalent to placing rooks on an (n-1,1) board, which can be done in R(n-1,1) ways. If there is no rook in row 2 of column 2, we position rooks so that exactly one of them lies in each of the n rows and n columns and the first two squares of the diagonal are empty. This corresponds to placing rooks on an (n,2) board, which can be done in R(n,2) ways. Combining the two groups of rook placements shows that R(n,1) = R(n-1,1) + R(n,2). Rewriting this as R(n,2) = R(n,1) – R(n-1,1) and substituting from (3) gives

$$R(n,2) = n! - (n-1)! - [(n-1)! - (n-2)!]$$
$$= n! - 2(n-1)! + (n-2)!$$

3. Divide the R(n,2) ways to place rooks on an (n,2) board into two groups, depending on whether or not there is a rook in row 3 of column 3. If there is, position other rooks so that exactly one of them lies in each of the remaining n – 1 rows and n – 1 columns and the first two squares on the diagonal are empty. This corresponds to placing rooks on an (n-1,2) board, which can be done in R(n-1,2) ways. On the other hand, if there is no rook in row 3 of column 3, position rooks so that exactly one of them lies in each of the n rows and n columns and the first three squares of the diagonal are empty. This corresponds to placing rooks on an (n,3) board, which can be done in R(n,3) ways. Combining the two groups of rook placements shows that R(n,2) = R(n-1,2) + R(n,3). Rewriting this as R(n,3) = R(n,2) – R(n-1,2) and substituting from (4) shows that

$$R(n,3) = n! - 2(n-1)! + (n-2)! - [(n-1)! - 2(n-2)! + (n-3)!]$$

$$= n! - 3(n-1)! + 3(n-2)! - (n-3)!$$

4. Row 6: 1 6 15 20 15 6 1

   Row 7: 1 7 21 35 35 21 7 1

5. $R(5,5) = 5! - 5(4!) + 10(3!) - 10(2!) + 5(1!) - 0!$

   $= 120 - 5(24) + 10(6) - 10(2) + 5(1) - 1$

   $= 120 - 120 + 60 - 20 + 5 - 1$

   $= 44$

   $R(7,5) = 7! - 5(6!) + 10(5!) - 10(4!) + 5(3!) - 2!$

   $= 5040 - 5(720) + 10(120) - 10(24) + 5(6) - 2$

   $= 5040 - 3600 + 1200 - 240 + 30 - 2$

   $= 2428$

6. $R(6,6) = 6! - 6(5!) + 15(4!) - 20(3!) + 15(2!) - 6(1!) + 0!$

   $= 720 - 6(120) + 15(24) - 20(6) + 15(2) - 6(1) + 1$

   $= 265$

7. Associate the ways to position rooks on a (6,0) board with the ways to hand out the papers of 6 students: a rook in row i of column j means that student i gets the paper of student j. Putting one rook in each row means that each student receives one paper. Putting one rook in each column means that each student's paper is handed out. Thus, the total number of ways to hand out one paper to each student is the number R(6,0) of ways to place rooks on a (6,0) board. By Equation 2, this number is 6! = 720.

   Each student receives someone else's paper when the rooks are positioned off the diagonal. The number of ways to do this is the number R(6,6) of ways to place rooks on a

(6,6) board. This number is 265, by Problem 6. Since the papers are handed out at random, the probability that each student receives someone else's paper is the fraction of all ways to hand out the papers that meet this condition. By the last two paragraphs, this fraction is 265/720 = 53/144.

8. Divide the R(n,k-1) ways to place rooks on an (n,k-1) board into two groups, depending on whether or not there is a rook in row k of column k. If there is, position other rooks so that exactly one of them lies in each of the remaining n – 1 rows and n – 1 columns and the first k – 1 squares of the diagonal are empty. This corresponds to placing rooks on an (n-1,k-1) board, and we can do it in R(n-1,k-1) ways. On the other hand, if there is no rook in row k of column k, position rooks so that exactly one of them lies in each of the n rows and n columns and the first k squares of the diagonal are empty. This corresponds to placing rooks on an (n,k) board, and we can do it in R(n,k) ways. combining the two groups of rook placements shows that R(n,k-1) = R(n-1,k-1) + R(n,k).

9. Problem 8 shows that

$$R(n,k) = R(n,k-1) - R(n-1,k-1).$$

Replace R(n,k-1) with (12) and R(n-1,k-1) with the result of substituting n-1 for n in (12). Then R(n,k) equals

$$\binom{k-1}{0}n! - \binom{k-1}{1}(n-1)! + \cdots + (-1)^{k-1}\binom{k-1}{k-1}(n-k+1)!$$

$$-\left[\binom{k-1}{0}(n-1)! - \binom{k-1}{1}(n-2)! + \cdots + (-1)^{k-1}\binom{k-1}{k-1}(n-k)!\right]. \quad (*)$$

We simplify this to (13).

Let s be an integer from 1 through k – 1. R(n,k-1) has a term $(-1)^s\binom{k-1}{s}(n-s)!$, and

R(n-1,k-1) has a term $(-1)^{s-1}\binom{k-1}{s-1}((n-1)-(s-1))!$. When we subtract the second term from the first and note that $-(-1)^{s-1} = (-1)^s$ and (n-1) – (s-1) = n – s, we get

$$(-1)^s\binom{k-1}{s}(n-s)! + (-1)^s\binom{k-1}{s-1}(n-s)!.$$

Factoring $(-1)^s(n-s)!$ out of both terms leaves $\binom{k-1}{s}+\binom{k-1}{s-1}$, which equals $\binom{k}{s}$, by (11).

Thus (*) has the same term $(-1)^s\binom{k}{s}(n-s)!$ as (13).

The previous paragraph matches up all terms of (*) and (13) except for the first and last terms of each. By (10), the first terms of both (*) and (13) are n!, and the last terms of both are $(-1)^k(n-k)!$. Thus, (*) equals (13).

Postscript

For k = n, Equation 8 says that R(n,n) is the sum of the terms $(-1)^s\binom{n}{s}(n-s)!$ for s running from 0 through n. When algebra books discuss combinations and the binomial theorem, they show that $\binom{n}{s}=\dfrac{n!}{s!(n-s)!}$.

Thus, the terms of R(n,n) have simplified form $(-1)^s\dfrac{n!}{s!}$. Factoring n! out of each term and noting that the terms for s = 0 and s = 1 cancel shows that

$$R(n,n) = n!\left(\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\cdots+(-1)^n\frac{1}{n!}\right)$$

for $n \geq 2$. This is an important formula for the number of ways to arrange n objects so that none of them stays in its original position. Such arrangments are called “derangements”.

Accordingly,

$$\frac{R(n,n)}{n!} = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots + (-1)^n \frac{1}{n!} \qquad (**)$$

is the probability that every student receives someone else’s paper when papers are handed out randomly to a class of n students. As n increases through the positive integers, this probability approaches 1/e = .3678… . In fact, calculus books show that

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

for all real numbers x, and setting x = -1 gives the formula

$$e^{-1} = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots.$$

Taking n = 6 in (**) shows that

$$\frac{R(6,6)}{6!} = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} = \frac{53}{144} = .3680...$$

(as in Problem 7), which is already close to 1/e.