

2010 Team Essay Solutions

Tournament Kings

1. A is a king because it beats B and D, and B beats C and E. B is a king because it beats C, D, and E, and E beats A. C is king because it beats A and D, A beats B, and D beats E. D is not a king because it beats only E, which loses to B. E is a king because it beats A and C, and A beats B and D. Thus the tournament has exactly four kings –A, B, C, and E.
2. There are multiple solutions. Here is one.

As in Figure 3 below, start with Figure 2 and add a dot F, arrows from F to B and C, and arrows to F from A, D, and E. A is a king because it beats B, D, and F, and B beats C and E. B is a king because it beats C, D, and E, and E beats A and F. C is a king because it beats A and D, A beats B and F, and D beats E. D is a king because it beats E and F, E beats A, and F beats B and C. E is a king because it beats A, C, and F, and A beats B and D. F is a king because it beats B and C, B beats D and E, and C beats A. Thus, all six teams A-F in the tournament are kings.

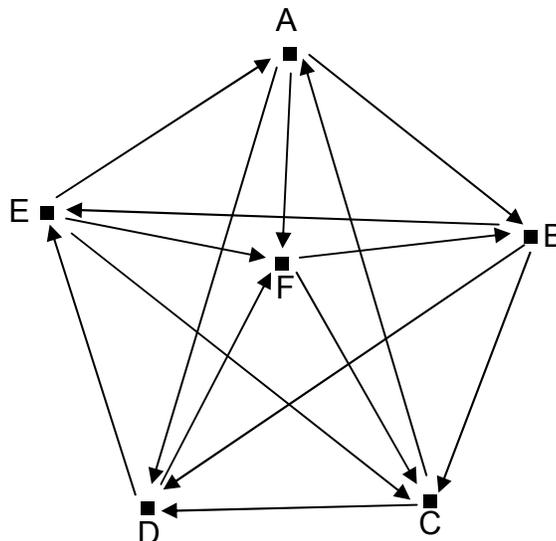


Figure 3

3. Let A be any one of the k original teams. For any other original team B , A beats B or an original team that beats B , since A is a king of the original tournament. Since A also beats X , and X beats Y , A is a king of the enlarged tournament. X is a king of the enlarged tournament because it beats Y , and Y beats all of the original teams. Y is a king of the enlarged tournament because it beats all of the original teams, which each beat X . Thus, all $k + 2$ teams of the enlarged tournament are kings.
4. Figure 4 below shows a three-team tournament where all teams are kings: each team beats the next team clockwise, which beats the third team. Then Problem 3 gives a five-team tournament where all teams are kings. By Problem 3 again, there is a seven-team tournament with all teams kings. Continuing in this way shows that, for every odd integer $k \geq 3$, there is a tournament with k teams where all teams are kings.

By Problem 2, there is a six-team tournament where all teams are kings. Then Problem 3 gives an eight-team tournament with all teams kings. By Problem 3 again, there is a ten-team tournament with all teams kings. Continuing in this way shows that, for every even integer $k \geq 6$, there is a tournament with k teams where all teams are kings.

Combining the last two paragraphs shows that there is a tournament with k teams where all k teams are kings for each integer $k \geq 3$ except 4.

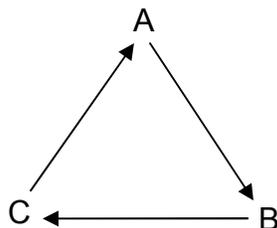


Figure 4

5. Each team plays three games. Since no team wins three or loses three games, each team wins one or two games. Since six games are played in all, the four teams have a total of six wins. Thus, two teams win two games apiece, and the other two teams win one game apiece.

Let A and D be the two teams that win two games apiece. Since A and D play each other, we can label them so that D beats A. Since D wins one other game, label the two remaining teams B and C so that D beats B and loses to C. A beats both B and C, since it wins two games and loses to D. Since B wins one game and loses to both A and D, it beats C. This gives Figure 1.

6. Consider any tournament with four teams. If some team A wins three games, then no other team is a king, since no other team beats A or a team that beats A. If some team D loses three games, then D is not a king since, for any other team Y, D doesn't beat Y or any team that beats Y. Finally, if no team wins three or loses three games, the teams can be labeled as in Figure 1 (by Problem 5), and the tournament has only three kings (by the paragraph before Problem 1). Thus, no tournament with four teams has all four teams as kings.
7. Let X be any team in the tournament other than P. X doesn't beat P and all the teams P beats, since X doesn't win more games than P. In other words, P beats either X or a team that beats X. Since this holds for all teams X other than P, P is a king.

8. Suppose first that Q loses to only one team P. Then P is a king because it beats Q and Q beats every team except itself and P.

On the other hand, suppose that Q loses at least twice. Consider the tournament formed by the teams that beat Q and the games they play with each other. This smaller tournament has a king P, by Problem 7.

We claim that P is a king of the original, larger tournament. In fact, P beats Q (since P is in the smaller tournament). If X is any team besides Q that isn't in the smaller tournament, then P beats Q and Q beats X (by definition of the smaller tournament). If X is any team besides P in the smaller tournament, then P beats X or a team that beats X (since P is a king of the smaller tournament). Thus, P is a king of the original, larger tournament, and P beats Q.

9. Q loses at least once because R beats a team that beats Q (since R is a king). Then the teams that beat Q include at least one king P, by Problem 8.

The king P is different from both Q and R, since P beats Q and Q beats R.

10. Let S be a tournament with m teams and k kings. Let n be an integer greater than m. Create a tournament T by starting with S, adding $n - m$ new teams, having each of the old teams beat each new team, and choosing any results for games between new teams.

The k kings of S remain kings in T since they beat all new teams. No new team A is a king of T because A doesn't beat any old team or any team that beats an old team (since A beats only new teams, which lose to all old teams). If B is an old team that is not a king in S, there is an old team C such that B does not beat C or any old team that beats C; since no new team beats C, B is still not a king in T. In short, T has exactly the same k kings as S.

11. No tournament has 0 kings (by Problem 7) or 2 (by Problem 9). For each nonnegative integer k other than 0 and 2, let m be the least number of teams in a tournament with k kings. Of course, we have $m \geq k$, since no tournament can have more kings than teams.

If $k = 1$, we have $m = 2$, since Figure 5 shows a tournament with 1 king and 2 teams, and every tournament has at least 2 teams. If $k = 3$ or $k \geq 5$, we have $m = k$, by Problem 4. If $k = 4$, we have $m = 5$, by Problems 6 and 1.

For each of the pairs of values of k and m in the previous paragraph, there is a tournament with k kings and n teams for all integers $n \geq m$, by Problem 10. Summarizing, there are tournaments with k kings and n teams for exactly those pairs of integers k and n with $n \geq 2$, $1 \leq k \leq n$, $k \neq 2$, and k and n not both 4.

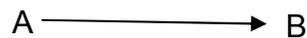


Figure 5